

LOCAL ZETA FUNCTIONS, PSEUDODIFFERENTIAL OPERATORS, AND SOBOLEV-TYPE SPACES OVER NON-ARCHIMEDEAN LOCAL FIELDS

W. A. ZÚÑIGA-GALINDO

ABSTRACT. In this article we introduce a new type of local zeta functions and study some connections with pseudodifferential operators in the framework of non-Archimedean fields. The new local zeta functions are defined by integrating complex powers of norms of polynomials multiplied by infinitely pseudo-differentiable functions. In characteristic zero, the new local zeta functions admit meromorphic continuations to the whole complex plane, but they are not rational functions. The real parts of the possible poles have a description similar to the poles of Archimedean zeta functions. But they can be irrational real numbers while in the classical case are rational numbers. We also study, in arbitrary characteristic, certain connections between local zeta functions and the existence of fundamental solutions for pseudodifferential equations.

1. INTRODUCTION

This article aims to explore the connections between local zeta functions (also called Igusa's local zeta functions) and pseudodifferential operators in the framework of the non-Archimedean fields. The local zeta functions over local fields, i.e. \mathbb{R} , \mathbb{C} , \mathbb{Q}_p , $\mathbb{F}_p((T))$, are ubiquitous objects in mathematics and mathematical physics, this is due mainly to the fact that they are the 'dual objects' of oscillatory integrals with analytic phases, see e.g. [2], [3], [6], [7], [9], [10], [11], [12], [16], [17], [18], [21], [25], [27], [28], [30], [33], [34], [35], [36] and the references therein. Let $(K, |\cdot|_K)$ be a local field of arbitrary characteristic, $\phi : K^n \rightarrow \mathbb{C}$ a test function, $\mathfrak{f} \in K[x_1, \dots, x_n]$ and $|d^n x|_K$ a Haar measure on K^n . The simplest type of local zeta function is defined as

$$(1.1) \quad Z_\phi(s, \mathfrak{f}) = \int_{K^n \setminus \mathfrak{f}^{-1}(0)} \phi |\mathfrak{f}|_K^s |d^n x|_K \text{ for } s \in \mathbb{C}, \text{ with } \operatorname{Re}(s) > 0.$$

These objects are deeply connected with string and Feynman amplitudes. Let us mention that the works of Speer [25] and Bollini, Giambiagi and González Domínguez [7] on regularization of Feynman amplitudes in quantum field theory are based on the analytic continuation of distributions attached to complex powers of polynomial functions in the sense of Gel'fand and Shilov [12]. For connections with string amplitudes see e.g. [8] and the references therein. In the Archimedean setting, the local zeta functions were introduced in the 50's by Gel'fand and Shilov. The main motivation was that the meromorphic continuation of Archimedean local

2000 *Mathematics Subject Classification.* Primary 11S40, 47S10; Secondary 46E39, 47G10.

Key words and phrases. Local zeta functions, Sobolev-type spaces, pseudodifferential operators, fundamental solutions, non-Archimedean operator theory.

The author was partially supported by Conacyt Grant No. 250845.

zeta functions implies the existence of fundamental solutions for differential operators with constant coefficients. This result has a non-Archimedean counterpart. In [35], see also [33] and the references therein, the author noticed that the classical argument showing that the analytic continuation of local zeta functions implies the existence of fundamental solutions also works in non-Archimedean fields of characteristic zero, and that for particular polynomials the Gel'fand-Shilov method gives explicit formulas for fundamental solutions. In this article, we use methods of pseudodifferential operators to study non-Archimedean local zeta functions.

A pseudodifferential operator with 'polynomial symbol' $|\mathfrak{h}|_K$, $\mathfrak{h} \in K[\xi_1, \dots, \xi_n]$, is defined as $\mathbf{A}(\partial, \mathfrak{h})\phi = \mathcal{F}_{\xi \rightarrow x}^{-1}(|\mathfrak{h}|_K \mathcal{F}_{x \rightarrow \xi} \phi)$, where \mathcal{F} denotes the Fourier transform in the space of test functions. A theory of non-Archimedean pseudodifferential equations is emerging motivated by its connections with mathematical physics, see e.g. [1], [20], [29], [33] and the references therein. The space of test functions is not invariant under the action of pseudodifferential operators. We replace it with $\mathcal{H}_\infty \subset L^2$ a Sobolev-type space which is a nuclear countably Hilbert space in the sense of Gel'fand-Vilenkin. This type of spaces was studied by the author in [32].

In this article we study the following integrals:

$$(1.2) \quad Z_{\mathcal{F}(g)}(s, \mathfrak{f}) = \int_{K^n \setminus \mathfrak{f}^{-1}(0)} |\mathfrak{f}|_K^s \mathcal{F}(g) |d^n x|_K$$

for $s \in \mathbb{C}$, with $\operatorname{Re}(s) > 0$, and $g \in \mathcal{H}_\infty$. For instance, if $\|\xi\|_K = \max_i |\xi_i|_K$, $t > 0$, and $\alpha > 0$, then $g(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{-t\|\xi\|_K^\alpha}) \in \mathcal{H}_\infty$. This function is the 'fundamental solution' of the heat equation over K^n , see e.g. [20], [29], [33]. We study these local zeta functions in Section 4.1 for certain polynomials. It is interesting to mention that the complex counterparts of these integrals are related with relevant arithmetic matters, see e.g. [9] and the references therein. The space of test functions is embedded in \mathcal{H}_∞ , and since the Fourier transform is an isomorphism on this space, integrals of type (1.2) are generalizations of the classical non-Archimedean local zeta functions (1.1). In characteristic zero, by using resolution of singularities, we show that integrals $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ admit meromorphic continuations to the whole complex plane as \mathcal{H}_∞^* -valued functions, here \mathcal{H}_∞^* denotes the strong dual of \mathcal{H}_∞ , see Theorem 2. These meromorphic continuations are not rational functions of q^{-s} , see Section 4.1, and the description of the real parts of the possible poles resembles the case of the Archimedean zeta functions, but there are relevant differences. If $\{Ni, v_i\}_{i \in T}$ are the numerical data of an embedded resolution of singularities of the map $\mathfrak{f}: K^n \rightarrow K$, then the real parts of the possible poles of the meromorphic continuation of $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ belongs to the set $\cup_{i \in T} \frac{-(v_i + M_i)}{N_i}$, where each M_i is an 'arbitrary arithmetic progression of real numbers', see Theorem 2. In the real case all the M_i are just the set of non-negative integers, see e.g. [16], [18]. The Hironaka resolution of singularities theorem [19] allow us to reduce the study of $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ to the case in which \mathfrak{f} is a monomial, like in the classical case. But the study of these monomial integrals does not follow the classical pattern because there is no a simply description for the functions in \mathcal{H}_∞ . All our results about the meromorphic continuation for monomial integrals are valid in arbitrary characteristic, see Section 4.3.

The pseudodifferential operators $\mathbf{A}(\partial, \mathfrak{h})$ give rise to continuous operators from \mathcal{H}_∞ onto itself, and thus they have continuous adjoints, denoted as $\mathbf{A}^*(\partial, \mathfrak{h})$, from \mathcal{H}_∞^* onto itself. In this framework, $Z_{\mathcal{F}(\mathbf{A}(\partial, \mathfrak{h}_i)g)}(s, \mathfrak{f})$ defines a \mathcal{H}_∞^* -valued function

for s in the half-plane $\operatorname{Re}(s) > 0$. Then, for instance, it makes sense to ask if $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ satisfies a pseudodifferential equation of the form

$$\sum_{i=1}^D c_i(q^{-s}) Z_{\mathcal{F}(\mathbf{A}(\partial, \mathfrak{h}_i)g)}(s + k_i, \mathfrak{f}) = 0,$$

where the $c_i(s) \in \mathbb{C}(q^{-s})$ and the k_i are integers. We also study the existence of fundamental solutions, i.e. solutions for equations of the form $\mathbf{A}^*(\partial, \mathfrak{f})E = \delta$ in \mathcal{H}_∞^* , where δ denotes the Dirac distribution. We show, like in the real case, see for instance [3], [6], [16], that the existence of a fundamental solution is equivalent to the division problem: there exists E in \mathcal{H}_∞^* such that $\widehat{E}|_{\mathfrak{f}|_K} = 1$ almost everywhere, here \widehat{E} denotes the Fourier of E as a distribution, see Theorem 3. Finally, by using the Gel'fand-Shilov method of analytic continuation, we show that the existence of an analytic continuation for $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ implies the existence of a fundamental solution for operator $\mathbf{A}^*(\partial, \mathfrak{f})$, see Theorem 4. These results are valid in arbitrary characteristic.

Another important motivation for studying integrals $Z_{\mathcal{F}(g)}(s, \mathfrak{f})$ comes from the fact that existence of meromorphic continuations for local zeta functions in local fields of positive characteristic is an open and difficult problem, see e.g. [17], [34], [36] and the references therein. A natural and possible way to attack this problem is by developing a suitable theory of D -modules in the framework of non-Archimedean fields of arbitrary characteristic, which would allow us to use Bernstein's approach to establish the meromorphic continuation for local zeta functions in positive characteristic, see e.g. [6], [16]. Several theories of arithmetic-type D -modules on fields of arbitrary characteristic have been constructed, see e.g. [5], [22]. However, all these theories involved operators acting on functions from K^n into K , and the operators needed to study local zeta functions must act on functions from K^n into \mathbb{C} , thus, the only possibility is to use pseudodifferential operators. Our results suggest the existence of a theory of pseudodifferential D -modules à la Bernstein which could be used to establish the analytic continuation of local zeta functions in arbitrary characteristic.

2. FOURIER ANALYSIS ON NON-ARCHIMEDEAN LOCAL FIELDS: ESSENTIAL IDEAS

In this section we fix the notation and collect some basic definitions on Fourier analysis on non-Archimedean local fields that we will use through the article. For an in-depth exposition the reader may consult [1], [26], [29], [30].

2.1. Non-Archimedean local fields. Along this article K will denote a *non-Archimedean local field* of arbitrary characteristic unless otherwise stated. The associated absolute value of K is denoted as $|\cdot|_K$. The ring of integers of K is $R_K = \{x \in K; |x|_K \leq 1\}$, its unique maximal ideal is $P_K = \{x \in R_K; |x|_K < 1\} = \pi R_K$, where π is a fixed generator of P_K , typically called a *local uniformizing parameter* of K ; and $R_K^\times = \{x \in R_K; |x|_K = 1\}$ is the *group of units* of R_K . The *residue field* of K is $R_K/P_K \simeq \mathbb{F}_q$, the finite field with q elements, where q is a power of a prime number p . Let $\operatorname{ord} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ denote the valuation of K . We assume that for $x \in K^\times$, $|x|_K = q^{-\operatorname{ord}(x)}$, i.e. $|\cdot|_K$ is a normalized absolute value. Every non-Archimedean local field of characteristic zero is isomorphic (as a topological field) to a finite extension of the field of p -adic numbers \mathbb{Q}_p . And any non-Archimedean

local field of characteristic p is isomorphic to a finite extension of the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q , see e.g. [30].

We extend the norm $\|\cdot\|_K$ to K^n by taking

$$\|x\|_K := \max_{1 \leq i \leq n} |x_i|_K, \quad \text{for } x = (x_1, \dots, x_n) \in K^n.$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $\|x\|_K = p^{-\text{ord}(x)}$. The metric space $(K^n, \|\cdot\|_K)$ is a complete ultrametric space, which is a totally disconnected topological space. For $l \in \mathbb{Z}$, denote by $B_l^n(a) = \{x \in K^n; \|x - a\|_K \leq q^l\}$ the ball of radius q^l with center at $a = (a_1, \dots, a_n) \in K^n$, and take $B_l^n(0) := B_l^n$. Note that $B_l^n(a) = B_l(a_1) \times \dots \times B_l(a_n)$, where $B_l(a_i) := \{x \in K; |x - a_i|_K \leq q^l\}$ is the one-dimensional ball of radius q^l with center at $a_i \in K$. The ball B_0^n equals the product of n copies of $B_0 := R_K$, the ring of integers of K . For $l \in \mathbb{Z}$, denote by $S_l^n(a) = \{x \in K^n; \|x - a\|_K = q^l\}$ the sphere of radius q^l with center at $a = (a_1, \dots, a_n) \in K^n$, and take $S_l^n(0) := S_l^n$.

2.2. Some function spaces. A complex-valued function ϕ defined on K^n is called *locally constant* if for any $x \in K^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that $\phi(x + x') = \phi(x)$ for $x' \in B_{l(x)}^N$. A such function is called a *Bruhat-Schwartz function* (or a *test function*) if it has compact support. The \mathbb{C} -vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(K^n) := \mathcal{D}$. Let $\mathcal{D}'(K^n) := \mathcal{D}'$ denote the set of all continuous functionals (distributions) on \mathcal{D} .

Along this article, $|d^n x|_K$ will denote a Haar measure on K^n normalized so that $\int_{R_K^n} |d^n x|_K = 1$. Given $r \in [0, \infty)$, we denote by $L^r(K^n, |d^n x|_K) := L^r(K^n) = L^r$, the \mathbb{C} -vector space of all the complex valued functions g satisfying $\int_{K^n} |g(x)|^r |d^n x|_K < \infty$; $L^\infty(K^n, |d^n x|_K) := L^\infty(K^n) = L^\infty$ denotes the \mathbb{C} -vector space of all the complex valued functions g such that the essential supremum of $|g|$ is bounded. Let denote by $C(K^n, \mathbb{C}) := C$, the \mathbb{C} -vector space of all the complex valued functions which are continuous. Set

$$C_0(K^n) := \left\{ f : K^n \rightarrow \mathbb{C}; f \text{ is continuous and } \lim_{x \rightarrow \infty} f(x) = 0 \right\},$$

where $\lim_{x \rightarrow \infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists a compact subset $B(\epsilon)$ such that $|f(x)| < \epsilon$ for $x \in K^n \setminus B(\epsilon)$. We recall that $(C_0(K^n), \|\cdot\|_{L^\infty})$ is a Banach space.

2.3. Fourier transform. We denote by $\chi(\cdot)$ a fixed additive character on K , i.e. a continuous map from K into the unit circle satisfying $\chi(y_0 + y_1) = \chi(y_0)\chi(y_1)$, $y_0, y_1 \in K$. If $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, we set $x \cdot \xi = \sum_{i=1}^n x_i \xi_i$.

If $g \in L^1$ its Fourier transform is defined by

$$(\mathcal{F}g)(\xi) = \int_{K^n} g(x) \chi(-x \cdot \xi) |d^n x|_K = \int_{K^n} g(x) \overline{\chi(x \cdot \xi)} |d^n x|_K,$$

where the bar denotes the complex conjugate. The Fourier transform is an isomorphism of \mathbb{C} -vector spaces from $\mathcal{D}(K^n)$ into itself satisfying

$$(2.1) \quad (\mathcal{F}(\mathcal{F}g))(x) = g(-x)$$

for every g in $\mathcal{D}(K^n)$. If $g \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}g)(\xi) = \lim_{l \rightarrow \infty} \int_{\|x\|_K \leq q^l} g(x) \chi(-x \cdot \xi) |d^n x|_K,$$

where the limit is taken in L^2 . We recall that the Fourier transform is unitary on L^2 , i.e. $\|g\|_{L^2} = \|\mathcal{F}g\|_{L^2}$ for $g \in L^2$ and that (2.1) is also valid in L^2 , see e.g. [26, Chapter III, Section 2]. We will also use the notation $\mathcal{F}_{x \rightarrow \xi}g$ and \widehat{g} for the Fourier transform of g .

The Fourier transform $\mathcal{F}(T)$ of a distribution $T \in \mathcal{D}'(K^n)$ is defined by

$$(\mathcal{F}(T), g) = (T, \mathcal{F}(g)) \text{ for all } g \in \mathcal{D}'(K^n).$$

The Fourier transform $T \rightarrow \mathcal{F}(T)$ is a linear isomorphism from $\mathcal{D}'(K^n)$ onto itself. Furthermore, $T = \mathcal{F}[\mathcal{F}(T)(-\xi)]$. We also use the notation $\mathcal{F}_{x \rightarrow \xi}T$ and \widehat{T} for the Fourier transform of T .

3. THE SPACES \mathcal{H}_∞

The Bruhat-Schwartz space $\mathcal{D}(K^n)$ is not invariant under the action of pseudo-differential operators. In this section, we review and expand some results about a class of nuclear countably Hilbert spaces introduced by the author in [32], these spaces are invariant under the action of large class of pseudodifferential operators. The notation here is slightly different to the notation used in [32], in addition, the results in [32] were formulated for \mathbb{Q}_p^n , but these results are valid in non-Archimedean local fields of arbitrary characteristic. For an in-depth discussion about nuclear countably Hilbert spaces, the reader may consult [13], [14], [15], [23].

Notation 1. We set $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. We denote by \mathbb{N} the set of non-negative integers. We set $[\xi]_K := \max(1, \|\xi\|_K)$.

We define for φ, ϱ in $\mathcal{D}(K^n)$ the following scalar product:

$$(3.1) \quad \langle \varphi, \varrho \rangle_l := \int_{K^n} [\xi]_K^l \widehat{\varphi}(\xi) \overline{\widehat{\varrho}(\xi)} |d^n \xi|_K,$$

for $l \in \mathbb{N}$, where the bar denotes the complex conjugate. We also set $\|\varphi\|_l^2 = \langle \varphi, \varphi \rangle_l$. Notice that $\|\cdot\|_l \leq \|\cdot\|_m$ for $l \leq m$. Let denote by $\mathcal{H}_l(K^n) := \mathcal{H}_l$ the completion of $\mathcal{D}(K^n)$ with respect to $\langle \cdot, \cdot \rangle_l$. Then $\mathcal{H}_m \hookrightarrow \mathcal{H}_l$ (continuous embedding) for $l \leq m$. We set

$$\mathcal{H}_\infty(K^n) := \mathcal{H}_\infty = \bigcap_{l \in \mathbb{N}} \mathcal{H}_l.$$

Notice that $\mathcal{H}_0 = L^2$ and that $\mathcal{H}_\infty \subset L^2$. With the topology induced by the family of seminorms $\|\cdot\|_{l \in \mathbb{N}}$, \mathcal{H}_∞ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right\}, \text{ with } f, g \in \mathcal{H}_\infty,$$

is a metric for the topology of \mathcal{H}_∞ considered as a convex topological space. A sequence $\{f_l\}_{l \in \mathbb{N}}$ in (\mathcal{H}_∞, d) converges to $f \in \mathcal{H}_\infty$, if and only if, $\{f_l\}_{l \in \mathbb{N}}$ converges to f in the norm $\|\cdot\|_l$ for all $l \in \mathbb{N}$. From this observation follows that the topology on \mathcal{H}_∞ coincides with the projective limit topology τ_P . An open neighborhood base at zero of τ_P is given by the choice of $\epsilon > 0$ and $l \in \mathbb{N}$, and the set

$$U_{\epsilon, l} := \{f \in \mathcal{H}_\infty; \|f\|_l < \epsilon\}.$$

The space \mathcal{H}_∞ endowed with the topology τ_P is a countably Hilbert space in the sense of Gel'fand and Vilenkin, see e.g. [14, Chapter I, Section 3.1] or [23, Section 1.2]. Furthermore $(\mathcal{H}_\infty, \tau_P)$ is metrizable and complete and hence a Fréchet space, cf. Lemma [32, Lemma 3.3]. In addition, the completion of the metric space

$(\mathcal{D}(K^n), d)$ is (\mathcal{H}_∞, d) , and this space is a nuclear countably Hilbert space, cf. [32, Lemma 3.4, Theorem 3.6].

Lemma 1. *With the above notation, the following assertions hold:*

- (i) $\mathcal{H}_\infty(K^n)$ is continuously embedded in $C_0(K^n)$;
- (ii) $\mathcal{H}_l(K^n) = \{f \in L^2(K^n); \|f\|_l < \infty\} = \{T \in \mathcal{D}'(K^n); \|T\|_l < \infty\}$;
- (iii) $\mathcal{H}_\infty(K^n) = \{f \in L^2(K^n); \|f\|_l < \infty, \text{ for every } l \in \mathbb{N}\}$;
- (iv) $\mathcal{H}_\infty(K^n) = \{T \in \mathcal{D}'(K^n); \|T\|_l < \infty, \text{ for every } l \in \mathbb{N}\}$. The equalities in (ii)-(iv) are in the sense of vector spaces.
- (v) $\mathcal{H}_\infty(K^n) \subset L^1(K^n)$. In particular, $\widehat{g} \in C_0(K^n)$ for $g \in \mathcal{H}_\infty(K^n)$.

Proof. (i) Take $f \in \mathcal{H}_\infty$ and $l > n$, then by using Cauchy-Schwarz inequality,

$$\begin{aligned} \|\widehat{f}\|_{L^1} &= \int_{K^n} |\widehat{f}| |d^n \xi|_K = \int_{K^n} \left\{ [\xi]_K^{\frac{1}{2}} |\widehat{f}| \right\} \left\{ \frac{1}{[\xi]_K^{\frac{1}{2}}} \right\} |d^n \xi|_K \\ &\leq \left\| \frac{1}{[\xi]_K^{\frac{1}{2}}} \right\|_{L^2} \left\| [\xi]_K^{\frac{1}{2}} |\widehat{f}| \right\|_{L^2} \leq C(n, l) \|f\|_l, \end{aligned}$$

where $C(n, l)$ is a positive constant, which shows that $\widehat{f} \in L^1$. Then, f is continuous and by the Riemann-Lebesgue theorem, (see e.g. [26, Theorem 1.6]), $f \in C_0(K^n)$. On the other hand, $\|f\|_{L^\infty} \leq \|\widehat{f}\|_{L^1} \leq C(n, l) \|f\|_l$, which shows that \mathcal{H}_l is continuously embedded in $C_0(K^n)$ for $l > n$. Thus $\mathcal{H}_\infty \subset C_0(K^n)$. Now, if $f_m \xrightarrow{d} f$ in $\mathcal{H}_\infty(K^n)$, i.e. if $f_m \xrightarrow{\|\cdot\|_l} f$ in \mathcal{H}_l for any $l \in \mathbb{N}$, then $f_m \xrightarrow{\|\cdot\|_{L^\infty}} f$ in $C_0(K^n)$.

(ii) In order to prove the first equality, it is sufficient to show that if $f \in L^2$ and $\|f\|_l < \infty$ then $f \in \mathcal{H}_l$. The condition $\|f\|_l < \infty$ is equivalent to $[\xi]_K^{\frac{1}{2}} \widehat{f} \in L^2(K^n)$, which implies that $\widehat{\{[\xi]_K^{\frac{1}{2}} \widehat{f}\}}(-\xi) \in L^2(K^n)$. By the density of $\mathcal{D}(K^n)$ in $L^2(K^n)$, there is a sequence $\{g_k\}_{k \in \mathbb{N}}$ in $\mathcal{D}(K^n)$ such that $g_k(\xi) \xrightarrow{\|\cdot\|_{L^2}} \widehat{\{[\xi]_K^{\frac{1}{2}} \widehat{f}\}}(-\xi)$, which implies that $\widehat{g}_k \xrightarrow{\|\cdot\|_{L^2}} [\xi]_K^{\frac{1}{2}} \widehat{f}$, which is equivalent to $\mathcal{F}^{-1}(\widehat{g}_k / [\xi]_K^{\frac{1}{2}}) \xrightarrow{\|\cdot\|_l} f$ with $\widehat{g}_k / [\xi]_K^{\frac{1}{2}} \in \mathcal{D}(K^n)$ for any $k \in \mathbb{N}$. To establish the second equality, we note that since $\|\cdot\|_0 \leq \|\cdot\|_l$ for any $l \in \mathbb{N}$, if $T \in \mathcal{H}_l$ then $\widehat{T} \in L^2$, and thus $T \in \mathcal{D}'(K^n)$ and $\|T\|_l < \infty$. Conversely, if $T \in \mathcal{D}'(K^n)$ and $\|T\|_l < \infty$ then $T \in L^2$ and $\|T\|_l < \infty$.

(iii) It follows from (ii).

(iv) It follows from (iii) by using that the following assertions are equivalent: (1) $T \in \mathcal{D}'(K^n)$ and $\|T\|_l < \infty$ for any $l \in \mathbb{N}$; (2) $T \in L^2$ and $\|T\|_l < \infty$ for any $l \in \mathbb{N}$.

(iv) By Theorem 3.15-(ii) in [32], $\mathcal{H}_\infty(K^n) \subset L^1(K^n)$. The fact that $\widehat{g} \in C_0(K^n)$ for $g \in \mathcal{H}_\infty(K^n)$ follows from the Riemann-Lebesgue theorem. \square

3.1. The dual space of \mathcal{H}_∞ . For $m \in \mathbb{N}$ and $T \in \mathcal{D}'(K^n)$, we set

$$\|T\|_{-m}^2 := \int_{K^n} [\xi]_K^{-m} \left| \widehat{T}(\xi) \right|^2 |d^n \xi|_K.$$

Then $\mathcal{H}_{-m} := \mathcal{H}_{-m}(K^n) = \{T \in \mathcal{D}'(K^n); \|T\|_{-m}^2 < \infty\}$ is a complex Hilbert space. Denote by \mathcal{H}_m^* the strong dual space of \mathcal{H}_m . It is useful to suppress the correspondence between \mathcal{H}_m^* and \mathcal{H}_m given by the Riesz theorem. Instead we

identify \mathcal{H}_m^* and \mathcal{H}_{-m} by associating $T \in \mathcal{H}_{-m}$ with the functional on \mathcal{H}_m given by

$$(3.2) \quad [T, g] := \int_{K^n} \widehat{T}(\xi) \widehat{g}(\xi) |d^n \xi|_K.$$

Notice that

$$(3.3) \quad |[T, g]| \leq \|T\|_{-m} \|g\|_m.$$

Now by a well-known result in the theory of countable Hilbert spaces, see e.g. [14, Chapter I, Section 3.1], $\mathcal{H}_0^* \subset \mathcal{H}_1^* \subset \dots \subset \mathcal{H}_m^* \subset \dots$ and

$$(3.4) \quad \mathcal{H}_\infty^*(K^n) = \mathcal{H}_\infty^* = \bigcup_{m \in \mathbb{N}} \mathcal{H}_{-m} = \{T \in \mathcal{D}'(K^n); \|T\|_{-l} < \infty, \text{ for some } l \in \mathbb{N}\}$$

as vector spaces. We mention that since \mathcal{H}_∞ is a nuclear space, cf. [32, Lemma 3.4, Theorem 3.6], the weak and strong convergence are equivalent in \mathcal{H}_∞^* , see e.g. [13, Chapter I, Section 6, Theorem 6.4]. We consider \mathcal{H}_∞^* endowed with the strong topology. On the other hand, let $B : \mathcal{H}_\infty^* \times \mathcal{H}_\infty \rightarrow \mathbb{C}$ be a bilinear functional. Then B is continuous in each of its arguments if and only if there exist norms $\|\cdot\|_m^{(a)}$ in \mathcal{H}_m^* and $\|\cdot\|_l^{(b)}$ in \mathcal{H}_l such that $|B(T, g)| \leq M \|T\|_m^{(a)} \|g\|_l^{(b)}$ with M a positive constant independent of T and g , see e.g. [14, Chapter I, Section 1.2] and [13, Chapter I, Section 4.1]. This implies that (3.2) is a continuous bilinear form on $\mathcal{H}_\infty^* \times \mathcal{H}_\infty$, which we will use as a pairing between \mathcal{H}_∞^* and \mathcal{H}_∞ .

Remark 1. The spaces $\mathcal{H}_\infty \subset L^2 \subset \mathcal{H}_\infty^*$ form a Gel'fand triple (also called a rigged Hilbert space), i.e. \mathcal{H}_∞ is a nuclear space which is densely and continuously embedded in L^2 and $\|g\|_{L^2}^2 = [g, g]$. This Gel'fand triple was introduced in [32].

Remark 2. By the proof of Lemma 1-(i), if $g \in \mathcal{H}_\infty$, then $\widehat{g} \in L^1 \cap L^2$ and by the dominated convergence theorem, $g(0) = \int \widehat{g} |d^n \xi|_K$. Consequently,

$$[\widehat{1}, g] = \int_{K^n} \widehat{g}(\xi) |d^n \xi|_K = g(0),$$

and thus $\widehat{1}$ defines an element of \mathcal{H}_∞^* , which we identify with the Dirac distribution δ , i.e. $[\delta, g] = g(0)$. In addition, $\delta * g = g$ for any $g \in \mathcal{H}_\infty$. Indeed, take $g_n \xrightarrow{\|\cdot\|_l} g$ for any $l \in \mathbb{N}$, with $\{g_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(K^n)$ and $g \in \mathcal{H}_\infty$. Then $\|\delta * g_n - g\|_l = \|g_n - g\|_l \rightarrow 0$, since $\delta * g_n = g_n$, for any $l \in \mathbb{N}$, which means that $g \rightarrow \delta * g$ is continuous in $\mathcal{D}(K^n)$, which is dense in \mathcal{H}_∞ .

3.2. Pseudodifferential operators acting on \mathcal{H}_∞ . Let \mathfrak{h}_i be a non-constant polynomial in $R_K[\xi_1, \dots, \xi_n]$ of degree d_i , for $i = 1, \dots, r$, with $1 \leq r \leq n$, and let α_i be a complex number such that $\operatorname{Re}(\alpha_i) > 0$ for $i = 1, \dots, r$. We set $\underline{\mathfrak{h}} = (\mathfrak{h}_1, \dots, \mathfrak{h}_r)$ and $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ and attach them the following pseudodifferential operator:

$$\begin{aligned} \mathcal{D}(K^n) &\rightarrow L^2 \cap C \\ \varphi &\rightarrow \mathbf{P}(\partial, \underline{\mathfrak{h}}, \underline{\alpha}) \varphi, \end{aligned}$$

where $(\mathbf{P}(\partial, \underline{\mathfrak{h}}, \underline{\alpha}) \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (\prod_{i=1}^r |\mathfrak{h}_i(\xi)|_K^{\alpha_i} \mathcal{F}_{x \rightarrow \xi} \varphi)$.

Notation 2. For $t \in \mathbb{R}$, we denote by $[t] := \min\{m \in \mathbb{Z}; m \geq t\}$, the ceiling function.

Lemma 2. *The mapping $P(\partial, \underline{h}, \alpha) : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is a well-defined continuous operator between locally convex spaces.*

Proof. The result follows from the following assertion:

Claim. $P(\partial, \underline{h}, \alpha) : \mathcal{H}_{m(l)} \rightarrow \mathcal{H}_l$, with $m(l) := l + 2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil$, defines a continuous operator between Banach spaces.

Indeed, by the Claim, if $g \in \mathcal{H}_\infty$, then $P(\partial, \underline{h}, \alpha) g \in \mathcal{H}_\infty$. To check the continuity, we take a sequence $\{g_k\}_{k \in \mathbb{N}}$ in \mathcal{H}_∞ such that $g_k \xrightarrow{d} g$, with $g \in \mathcal{H}_\infty$, i.e. $g_k \xrightarrow{\|\cdot\|_l} g$ for any $l \in \mathbb{N}$. By the Claim

$$\|P(\partial, \underline{h}, \alpha) g_k - P(\partial, \underline{h}, \alpha) g\|_l \leq \|g_k - g\|_{m(l)}.$$

which implies that $P(\partial, \underline{h}, \alpha) g_k \xrightarrow{\|\cdot\|_l} P(\partial, \underline{h}, \alpha) g$ for any $l \in \mathbb{N}$.

Proof of the Claim. By taking $\varphi \in \mathcal{D}(K^n)$, and using that

$$(3.5) \quad |\mathfrak{h}_i(\xi)|_K^{\operatorname{Re}(\alpha_i)} \leq [\xi]_K^{d_i \lceil \operatorname{Re}(\alpha_i) \rceil},$$

where d_i denotes the degree of \mathfrak{h}_i , we have

$$\begin{aligned} \|P(\partial, \underline{h}, \alpha) g\|_l^2 &= \int_{K^n} [\xi]^l \prod_{i=1}^r |\mathfrak{h}_i(\xi)|_K^{2 \operatorname{Re}(\alpha_i)} |\widehat{\varphi}(\xi)|^2 |d^n \xi|_K \\ &\leq \int_{K^n} [\xi]^{l+2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil} |\widehat{\varphi}(\xi)|^2 |d^n \xi|_K \leq \|\varphi\|_{l+2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil}^2. \end{aligned}$$

Now, from the density of $\mathcal{D}(K^n)$ in $\mathcal{H}_{l+2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil}$, we conclude that

$$P(\partial, \underline{h}, \alpha) : \mathcal{H}_{l+2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil} \rightarrow \mathcal{H}_l$$

defines a continuous operator between Banach spaces. \square

3.2.1. Adjoint operators on \mathcal{H}_∞ . By using that $P(\partial, \underline{h}, \alpha) : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is a continuous operator and some results on adjoint operators in the setting of locally convex spaces, see e.g. [31, Chapter VII, Section 1], one gets that there exists a continuous operator $P^*(\partial, \underline{h}, \alpha) : \mathcal{H}_\infty^* \rightarrow \mathcal{H}_\infty^*$ satisfying

$$[P^*(\partial, \underline{h}, \alpha) T, g] = [T, P(\partial, \underline{h}, \alpha) g]$$

for any $T \in \mathcal{H}_\infty^*$ and any $g \in \mathcal{H}_\infty$. We call P^* the *adjoint operator* of P .

3.3. Some additional results.

Lemma 3. *Take $\underline{h} = (\mathfrak{h}_1, \dots, \mathfrak{h}_r)$ with $\mathfrak{h}_i \in R_K[\xi_1, \dots, \xi_n] \setminus R_K$, $1 \leq r \leq n$, and $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{C}^r$ with $\operatorname{Re}(\alpha_i) > 0$ for any i , as before, and $g \in \mathcal{H}_\infty$, and define*

$$I_g(\alpha, \underline{h}) = \int_{K^n \setminus \bigcup_{i=1}^r \mathfrak{h}_i^{-1}(0)} \prod_{i=1}^r |\mathfrak{h}_i(\xi)|_K^{\alpha_i} \widehat{g}(\xi) |d^n \xi|_K.$$

Then $I_g(\alpha, \underline{h})$ defines a \mathcal{H}_∞^ -valued holomorphic function of α in the half-plane $\operatorname{Re}(\alpha) > 0$ for $i = 1, \dots, r$.*

Proof. By using (3.5), we get

$$|I_g(\alpha, \underline{h})| \leq \int_{K^n} [\xi]_K^{\sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil} |\widehat{g}(\xi)| |d^n \xi|_K,$$

and by proof of Lemma 1-(i),

$$|I_g(\alpha, \underline{h})| \leq C(n, l) \|g\|_{l+2}^2 \sum_{i=1}^r d_i \lceil \operatorname{Re}(\alpha_i) \rceil$$

for any positive integer $l > n$. This implies that $I_g(\alpha, \underline{h})$ is a \mathcal{H}_∞^* -valued function if α belongs to the half-plane $\operatorname{Re}(\alpha_i) > 0$, $i = 1, \dots, r$. To establish the holomorphy of $I_g(\alpha, \underline{h})$, we recall that a continuous complex-valued function defined in an open set $A \subseteq \mathbb{C}^r$, which is holomorphic in each variable separately, is holomorphic in A . Thus, it is sufficient to show that $I_g(\alpha, \underline{h})$ is holomorphic in each α_i . This last fact follows from a classical argument, see e.g. [16, Lemma 5.3.1], by showing the existence of a function $\Phi_K(\xi) \in L^1$ such that for any compact subset \mathcal{K} of $\{\alpha_i \in \mathbb{C}; \operatorname{Re}(\alpha_i) > 0\}$, with α_j fixed for $j \neq i$, it verifies that $\prod_{i=1}^r |\hat{h}_i(\xi)|_K^{\operatorname{Re}(\alpha_i)} |\hat{g}(\xi)| \leq \Phi_K(\xi)$ for $\alpha_i \in \mathcal{K}$. \square

Notation 3. If I is a finite set, then $|I|$ denotes its cardinality.

Lemma 4. Let I and J be two non-empty subsets of $\{1, \dots, n\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$. Set $x = (x_1, \dots, x_n) = (x_I, x_J) \in K^{|I|} \times K^{|J|}$ with $x_I = (x_i)_{i \in I}$ and $x_J = (x_i)_{i \in J}$. With this notation, the measure $|d^n x|_K$ becomes the product measure of $|d^{|I|} x_I|_K$ and $|d^{|J|} x_J|_K$. Fix $\xi_J^{(0)} \in K^{|J|}$. Then the mapping

$$P_{J, \xi_J^{(0)}} : \mathcal{H}_\infty(K^n) \rightarrow \mathcal{H}_\infty(K^{|I|})$$

$$g(x_I, x_J) \rightarrow \int_{K^{|J|}} \chi(x_J \cdot \xi_J^{(0)}) g(x_I, x_J) |d^{|J|} x_J|_K$$

gives rise to a well-defined linear continuous operator.

Proof. By using that $g(x_I, x_J) \in L^1(K^n, |d^n x|_K)$, cf. [32, Theorem 3.15-(ii)], and applying Fubini's theorem, $g(x_I, x_J) \in L^1(K^{|J|}, |d^{|J|} x_J|_K)$ for almost all the x_I 's. Thus

$$\mathcal{F}_{x_I \rightarrow \xi_I}^{x_J \rightarrow \xi_J} (g) \big|_{\xi_J = \xi_J^{(0)}} = \mathcal{F}_{x_I \rightarrow \xi_I} \left(P_{J, \xi_J^{(0)}} g \right).$$

Now,

$$\begin{aligned} \|P_{J, \xi_J^{(0)}} g\|_l^2 &= \int_{K^{|I|}} [\xi_I]_K^l \left| \mathcal{F}_{x_I \rightarrow \xi_I} \left(P_{J, \xi_J^{(0)}} g \right) \right|^2 |d^{|I|} \xi_I|_K \\ &= \int_{K^{|I|}} [\xi_I]_K^l \left| \hat{g}(\xi_I, \xi_J^{(0)}) \right|^2 |d^{|I|} \xi_I|_K \leq \int_{K^n} [\xi]_K^l |\hat{g}(\xi_I, \xi_J)|^2 |d^n \xi|_K \\ &= \|g\|_l^2 < \infty \end{aligned}$$

for any $l \in \mathbb{N}$, which implies that $P_{J, \xi_J^{(0)}}$ is a continuous operator. \square

Lemma 5. Fix two subsets I, J of $\{1, \dots, n\}$ satisfying $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$. Set $\alpha_I = (\alpha_i)_{i \in I} \in \mathbb{C}^{|I|}$ and $\beta_J = (\beta_i)_{i \in J} \in \mathbb{C}^{|J|}$. Assume that $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$ and $\operatorname{Re}(\beta_i) > 0$ for $i \in J$. Set for $g \in \mathcal{H}_\infty$,

$$E_{\hat{g}}(\alpha_I, \beta_J) := \int_{K^n \setminus \{0\}} \frac{\prod_{i \in I} |\xi_i|_K^{\alpha_i}}{\prod_{i \in J} |\xi_i|_K^{\beta_i}} \hat{g}(\xi) |d^n \xi|_K,$$

with the convention that $\prod_{i \in \emptyset} = 1$. Then $E_{\hat{g}}(\alpha_I, \beta_J)$ gives rise to a \mathcal{H}_∞^* -valued function which is holomorphic in α_I and β_J in the open set $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$ and $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in J$.

Proof. We first consider the case $J = \emptyset$,

$$E_{\widehat{g}}(\alpha_I) = \int_{K^n \setminus \{0\}} \prod_{i \in I} |\xi_i|_K^{\alpha_i} \widehat{g}(\xi) |d^n \xi|_K.$$

By Lemma 3, $E_{\widehat{g}}(\alpha_I)$ defines a \mathcal{H}_{∞}^* -valued function which is holomorphic in α_I in the open set $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$.

We assume that $J \neq \emptyset$. For $L \subseteq \{1, \dots, n\}$, we define

$$A_L = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n; |\xi_i|_K > 1 \Leftrightarrow i \in L\}.$$

Then

$$K^n = \bigsqcup_{L \subseteq \{1, \dots, n\}} A_L \text{ and } E_{\widehat{g}}(\alpha_I, \beta_J) = \sum_{L \subseteq \{1, \dots, n\}} E_{\widehat{g}}^{(L)}(\alpha_I, \beta_J)$$

where

$$E_{\widehat{g}}^{(L)}(\alpha_I, \beta_J) := \int_{A_L \setminus \{0\}} \frac{\prod_{i \in I} |\xi_i|_K^{\alpha_i}}{\prod_{i \in J} |\xi_i|_K^{\beta_i}} \widehat{g}(\xi) |d^n \xi|_K.$$

The proof is accomplished by showing that each functional $E_{\widehat{g}}^{(L)}(\alpha_I, \beta_J)$ satisfies the requirements announced for $E_{\widehat{g}}(\alpha_I, \beta_J)$.

Case 1. If $L = \emptyset$, $A_L = R_K^n$.

In this case, with the notation of Lemma 4, we have

$$\begin{aligned} E_{\widehat{g}}^{(L)}(\alpha_I, \beta_J) &= \int_{R_K^n \setminus \{0\}} \frac{\prod_{i \in I} |\xi_i|_K^{\alpha_i}}{\prod_{i \in J} |\xi_i|_K^{\beta_i}} \widehat{g}(\xi) |d^n \xi|_K \\ &= \int_{R_K^{|J|} \setminus \{0\}} \frac{1}{\prod_{i \in J} |\xi_i|_K^{\beta_i}} \left\{ \int_{R_K^{|I|}} \prod_{i \in I} |\xi_i|_K^{\alpha_i} \mathcal{F}_{x_I \rightarrow \xi_I}(\mathbf{P}_{J, \xi_J} g) \left| d^{|I|} \xi_I \right|_K \right\} \left| d^{|J|} \xi_J \right|_K, \end{aligned}$$

and

$$\begin{aligned} \left| E_{\widehat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq \int_{R_K^{|J|} \setminus \{0\}} \frac{1}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \times \\ &\quad \left\{ \int_{R_K^{|I|}} |\mathcal{F}_{x_I \rightarrow \xi_I}(\mathbf{P}_{J, \xi_J} g)| \left| d^{|I|} \xi_I \right|_K \right\} \left| d^{|J|} \xi_J \right|_K \\ &\leq \left\{ \int_{R_K^{|J|} \setminus \{0\}} \frac{\left| d^{|J|} \xi_J \right|_K}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \sup_{\xi_J \in K^{|J|}} \int_{R_K^{|I|}} |\mathcal{F}_{x_I \rightarrow \xi_I}(\mathbf{P}_{J, \xi_J} g)| \left| d^{|I|} \xi_I \right|_K \\ &\leq \left\{ \int_{R_K^{|J|} \setminus \{0\}} \frac{\left| d^{|J|} \xi_J \right|_K}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \sup_{\xi_J \in K^{|J|}} \int_{R_K^{|I|}} \frac{1}{[\xi_I]_K^{\frac{1}{2}}} [\xi_I]_K^{\frac{1}{2}} |\mathcal{F}_{x_I \rightarrow \xi_I}(\mathbf{P}_{J, \xi_J} g)| \left| d^{|I|} \xi_I \right|_K. \end{aligned}$$

Now, by applying Cauchy-Schwarz inequality and Lemma 4,

$$\begin{aligned}
\left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq \left\{ \int_{R_K^{|J|} \setminus \{0\}} \frac{|d^{|J|} \xi_J|_K}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \times \\
&\sup_{\xi_J \in K^{|J|}} \sqrt{\int_{R_K^{|I|}} \frac{|d^{|I|} \xi_I|_K}{[\xi_I]_K^l}} \sqrt{\int_{R_K^{|I|}} [\xi_I]_K^l |\mathcal{F}(\mathbf{P}_{J, \xi_J} g)|^2 |d^{|I|} \xi_I|_K} \\
&\leq \left\{ \int_{R_K^{|J|} \setminus \{0\}} \frac{|d^{|J|} \xi_J|_K}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \sup_{\xi_J \in K^{|J|}} \sqrt{\int_{K^n} [\xi]_K^l |\hat{g}(\xi)|^2 |d^n \xi|_K} \\
&\leq \left\{ \int_{R_K^{|J|} \setminus \{0\}} \frac{|d^{|J|} \xi_J|_K}{\prod_{i \in J} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \|g\|_l.
\end{aligned}$$

We now use that, if $0 < \operatorname{Re}(\beta_i) < 1$ then $\frac{1}{|\xi_i|_K^{\operatorname{Re}(\beta_i)}} \in L^1(R_K, |d\xi_i|_K)$, to conclude that

$$(3.6) \quad \left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| \leq C(\beta_J) \|g\|_l$$

for any $l \in \mathbb{N}$. This implies that $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ is a \mathcal{H}_{∞}^* -valued function for β_J in the set $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in J$ and $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$. In order to show that $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ is holomorphic in (α_I, β_J) in α_I, β_J in the set $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in J$ and $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$, we show that $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ is holomorphic in each variable separately. This fact is established by using (3.6) and Lemma 5.3.1 in [16].

Case 2. If $L = \{1, \dots, n\}$, $A_L = \{\xi \in K^n; |\xi_i|_K > 1 \text{ for } i = 1, \dots, n\}$.

In this case

$$\begin{aligned}
\left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq \int_{A_L} \prod_{i \in I} |\xi_i|_K^{\operatorname{Re}(\alpha_i)} |\hat{g}(\xi)| |d^n \xi|_K \\
&\leq \int_{A_L} [\xi]_K^{\sum_{i \in I} \lceil \operatorname{Re}(\alpha_i) \rceil} |\hat{g}(\xi)| |d^n \xi|_K \leq C(l, n) \|g\|_{l+2 \sum_{i \in I} \lceil \operatorname{Re}(\alpha_i) \rceil},
\end{aligned}$$

for any positive integer $l > n$, which implies that $E_{\hat{g}}^{(L)}(\alpha_I)$ is a \mathcal{H}_{∞}^* -valued function in the set $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$.

We now analyze the case where L is a non-empty and proper subset from $\{1, \dots, n\}$ and $J \neq \emptyset$.

Case 3. If $J \cap L = \emptyset$, i.e. $L \subseteq I$.

In this case, by proceeding as in the proof of Case 1, and using that

$$\int_{K^{|L|}} \frac{|d^{|L|} \xi_L|_K}{[\xi]_K^l} < \infty \text{ for } l > n,$$

one gets that

$$\left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| \leq C(\beta_J, l, n) \|g\|_{l+2 \sum_{i \in I} \lceil \operatorname{Re}(\alpha_i) \rceil},$$

which implies that $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ is a \mathcal{H}_{∞}^* -valued function for β_J in the set $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in J$ and $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$. The verification that $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ is holomorphic in α_I, β_J in the set $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in J$ and $\operatorname{Re}(\alpha_i) > 0$ for $i \in I$ is done like in Case 1.

Case 4. $J \cap L \neq \emptyset$ and $I \cap L = \emptyset$ (i.e. $L \subseteq J$) and $M := J \setminus L$.

In this case, taking $\{1, \dots, n\} = L \sqcup L'$ and

$$(3.7) \quad A_L \setminus \{0\} = \left(R_K^{|L'|} \setminus \{0\} \right) \sqcup \left(K^{|L|} \setminus \{0\} \right),$$

and using the notation of Lemma 4, $E_{\hat{g}}^{(L)}(\alpha_I, \beta_J)$ equals

$$\int_{R_K^{|L'|} \setminus \{0\}} \frac{1}{\prod_{i \in M} |\xi_i|_K^{\beta_i}} \left\{ \int_{K^{|L|} \setminus \{0\}} \frac{\prod_{i \in I} |\xi_i|_K^{\alpha_i}}{\prod_{i \in L} |\xi_i|_K^{\beta_i}} \mathcal{F}_{x_L \rightarrow \xi_L}(\mathbf{P}_{L', \xi_{L'}, g}) \left| d^{|L|} \xi_L \right|_K \right\} \left| d^{|L'|} \xi_{L'} \right|_K$$

and thus

$$\begin{aligned} \left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq \left\{ \int_{R_K^{|L'|} \setminus \{0\}} \frac{\left| d^{|L'|} \xi_{L'} \right|_K}{\prod_{i \in M} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \sup_{\xi_{L'} \in K^{|L'|}} \int_{K^{|L|}} \left| \mathcal{F}_{x_L \rightarrow \xi_L}(\mathbf{P}_{L', \xi_{L'}, g}) \right| \left| d^{|L|} \xi_L \right|_K \\ &= \left\{ \int_{R_K^{|L'|} \setminus \{0\}} \frac{\left| d^{|L'|} \xi_{L'} \right|_K}{\prod_{i \in M} |\xi_i|_K^{\operatorname{Re}(\beta_i)}} \right\} \times \\ &\quad \sup_{\xi_{L'} \in K^{|L'|}} \int_{K^{|L|}} \frac{1}{\left[\xi_L \right]_K^{\frac{l}{2}}} \left[\xi_L \right]_K^{\frac{l}{2}} \left| \mathcal{F}_{x_L \rightarrow \xi_L}(\mathbf{P}_{L', \xi_{L'}, g}) \right| \left| d^{|L|} \xi_L \right|_K. \end{aligned}$$

By taking $l > n$, $0 < \operatorname{Re}(\beta_i) < 1$ for $i \in M$, and applying Cauchy-Schwarz inequality and Lemma 4,

$$\begin{aligned} \left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq C(\beta_J, l, n) \times \\ &\quad \sup_{\xi_{L'} \in K^{|L'|}} \sqrt{\int_{K^{|L|}} \left[\xi_L \right]_K^l \left| \mathcal{F}_{x_L \rightarrow \xi_L}(\mathbf{P}_{L', \xi_{L'}, g}) \right|^2 \left| d^{|L|} \xi_L \right|_K} \leq C(\beta_J, l, n) \|g\|_l. \end{aligned}$$

Case 5. $J \cap L \neq \emptyset$, $I \cap L \neq \emptyset$, and $M := J \setminus L$.

In this case, taking $\{1, \dots, n\} = L \sqcup L'$ and using (3.7), and proceeding like in Case 4,

$$\begin{aligned} \left| E_{\hat{g}}^{(L)}(\alpha_I, \beta_J) \right| &\leq \left\{ \int_{R_K^{|M|} \setminus \{0\}} \frac{|d^{|M|} \xi_M|_K}{\prod_{i \in M} |\xi_i|_K^{\text{Re}(\beta_i)}} \right\} \times \\ &\sup_{\xi_{L'} \in K^{|L'|}} \int_{K^{|L|}} \prod_{i \in I \cap L} |\xi_i|_K^{\text{Re}(\alpha_i)} |\mathcal{F}_{x_{L'} \rightarrow \xi_{L'}}(\mathbf{P}_{L', \xi_{L'}, g})| |d^{|L|} \xi_L|_K \\ &\leq C(\beta_J, l, n) \|g\|_{l+2 \sum_{i \in I \cap L} [\text{Re}(\alpha_i)]} \end{aligned}$$

for $l > n$. \square

4. LOCAL ZETA FUNCTIONS IN \mathcal{H}_∞

We fix a non-constant polynomial \mathfrak{f} in $R_K[\xi_1, \dots, \xi_n]$ of degree d . We set $\widetilde{\mathfrak{f}}(\xi) := \mathfrak{f}(-\xi)$. Then $\widehat{\widetilde{\mathfrak{f}}}_K^{\overline{s}}$, $\text{Re}(s) > 0$, defines a distribution from \mathcal{D}' satisfying $\widehat{\widetilde{\mathfrak{f}}}_K^{\overline{s}} = |\mathfrak{f}|_K^{\overline{s}}$ in \mathcal{D}' for $\text{Re}(s) > 0$. In addition, $\widehat{\widetilde{\mathfrak{f}}}_K^{\overline{s}}$, with $\text{Re}(s) > 0$, gives rise to a holomorphic \mathcal{H}_∞^* -valued function in s , cf. Lemma 3. In this section we establish the existence of a meromorphic continuation of \mathcal{H}_∞^* -valued functions of type

$$(4.1) \quad Z_{\hat{g}}(s, \mathfrak{f}) := \left[\widehat{\widetilde{\mathfrak{f}}}_K^{\overline{s}}, g \right] = \int_{K^n \setminus \widetilde{\mathfrak{f}}^{-1}(0)} |\mathfrak{f}(\xi)|_K^s \widehat{g}(\xi) |d^n \xi|_K,$$

$\text{Re}(s) > 0$, $g \in \mathcal{H}_\infty(K^n)$, to the whole complex plane. Since $\mathcal{D}(K^n) \subset \mathcal{H}_\infty(K^n)$, integrals of type (4.1) are generalizations of the classical Igusa's local zeta functions, see e.g. [16], [18]. Before further discussion we present an example that illustrates the analogies and differences between the classical Igusa's zeta functions and the local zeta functions on \mathcal{H}_∞ .

4.1. Local zeta functions for strongly non-degenerate forms modulo π .

Notation 4. We denote by $\bar{\cdot}$, the reduction modulo π , i.e. the canonical mapping $R_K^n \rightarrow (R_K/\pi R_K)^n = \mathbb{F}_q^n$. If \mathfrak{f} is a polynomial with coefficients in R_K , we denote by $\bar{\mathfrak{f}}$, the polynomial obtained by reducing modulo π the coefficients of \mathfrak{f} .

We take $g(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}(e^{-\|\xi\|_K^\alpha})$, with $\alpha > 0$. By Lemma 1-(iii), $g \in \mathcal{H}_\infty(K^n)$. It is interesting to mention that function $\mathcal{F}_{\xi \rightarrow x}^{-1}(e^{-t\|\xi\|_K^\alpha})$ with $t > 0$, $\alpha > 0$ is the 'fundamental solution' of the heat equation over K^n , see e.g. [33, Section 2.2.7]. We also pick a homogeneous polynomial \mathfrak{f} with coefficients in $R_K \setminus \pi R_K$ of degree d , satisfying $\bar{\mathfrak{f}}(\bar{a}) = \nabla \bar{\mathfrak{f}}(\bar{a}) = \bar{0}$ implies $\bar{a} = \bar{0}$. We set

$$\left[\widehat{\widetilde{\mathfrak{f}}}_K^{\overline{s}}, g \right] = \int_{K^n \setminus \widetilde{\mathfrak{f}}^{-1}(0)} |\mathfrak{f}(\xi)|_K^s e^{-\|\xi\|_K^\alpha} |d^n \xi|_K \text{ for } \text{Re}(s) > 0.$$

Claim. $\left[\widehat{\left| \mathfrak{f} \right|_K^s}, g \right]$ admits meromorphic continuation to the whole complex plane and the real parts of the possible poles belong to the set $\{-1\} \cup \bigcup_{l \in \mathbb{N}} \left\{ \frac{-(n+\alpha l)}{d} \right\}$.

To establish the Claim we proceed as follows. We use the partition $K^n \setminus \{0\} = \bigsqcup_{j=-\infty}^{\infty} \pi^j S_0^n$ to obtain

$$\begin{aligned}
 (4.2) \quad \left[\widehat{\left| \mathfrak{f} \right|_K^s}, g \right] &= \sum_{j=-\infty}^{\infty} \int_{\pi^j S_0^n} |\mathfrak{f}(\xi)|_K^s e^{-\|\xi\|_K^\alpha} |d^n \xi|_K \\
 &= \left(\int_{S_0^n} |\mathfrak{f}(\xi)|_K^s |d^n \xi|_K \right) \left(\sum_{j=-\infty}^{\infty} q^{-jds-jn} e^{-q^{-j\alpha}} \right) \\
 &= \left(\int_{S_0^n} |\mathfrak{f}(\xi)|_K^s |d^n \xi|_K \right) \left(\frac{1}{1-q^{-n}} \int_{K^n} \|\xi\|_K^{ds} e^{-\|\xi\|_K^\alpha} |d^n \xi|_K \right) =: Z_0(s) Z_1(s).
 \end{aligned}$$

By using that

$$Z(s, \mathfrak{f}) := \int_{R_K^n} |\mathfrak{f}(\xi)|_K^s |d^n \xi|_K \frac{L(q^{-s})}{(1-q^{-1-s})(1-q^{-n-ds})} \text{ for } \operatorname{Re}(s) > 0,$$

where $L(q^{-s})$ is a polynomial in q^{-s} , see e.g. [16, Proposition 10.2.1], and the partition $R_K^n = \pi R_K^n \sqcup S_0^n$, we have $Z(s, \mathfrak{f}) = q^{-n-ds} Z(s, \mathfrak{f}) + Z_0(s)$, and consequently

$$(4.3) \quad Z_0(s) = \frac{L(q^{-s})}{(1-q^{-1-s})}.$$

Now

(4.4)

$$\begin{aligned}
 Z_1(s) &= \frac{1}{1-q^{-n}} \int_{R_K^n} \|\xi\|_K^{ds} e^{-\|\xi\|_K^\alpha} |d^n \xi|_K + \frac{1}{1-q^{-n}} \int_{K^n \setminus R_K^n} \|\xi\|_K^{ds} e^{-\|\xi\|_K^\alpha} |d^n \xi|_K \\
 &=: Z_{1,1}(s) + Z_{1,2}(s).
 \end{aligned}$$

The integral $Z_{1,2}(s)$ is holomorphic in the whole complex plane. To study $Z_{1,1}(s)$ we use $e^{-\|\xi\|_K^\alpha} = \sum_{l=0}^L \frac{(-1)^l}{l!} \|\xi\|_K^{\alpha l} + f(\|\xi\|_K)$ as follows:

(4.5)

$$\begin{aligned}
 Z_{1,1}(s) &= \frac{1}{1-q^{-n}} \sum_{l=0}^L \frac{(-1)^l}{l!} \int_{R_K^n} \|\xi\|_K^{ds+\alpha l} |d^n \xi|_K + \frac{1}{1-q^{-n}} \int_{R_K^n} \|\xi\|_K^{ds} f(\|\xi\|_K) |d^n \xi|_K \\
 &= \sum_{l=0}^L \frac{(-1)^l}{l!} \frac{1}{1-q^{-ds-n-\alpha l}} + \frac{1}{1-q^{-n}} \int_{R_K^n} \|\xi\|_K^{ds} f(\|\xi\|_K) |d^n \xi|_K.
 \end{aligned}$$

The announced Claim follows from (4.2)-(4.5). The local zeta functions $\left[\widehat{\left| \mathfrak{f} \right|_K^s}, g \right]$ admit meromorphic continuations to the whole complex plane but they are not

rational functions of q^{-s} and the real parts of the possible poles are ‘real’ negative numbers. In addition, if take $\alpha = 1$, then the real parts of the poles of the meromorphic continuations resemble the poles of Archimedean zeta functions, see e.g. [16, Theorem 5.4.1].

4.2. Multidimensional Vladimirov operators.

4.2.1. *Riesz kernels.* For $\alpha \in \mathbb{C}$, we set $\Gamma(\alpha) := \frac{1-q^{\alpha-1}}{1-q^{-\alpha}}$. The function

$$f_\alpha(x) = \frac{|x|_K^{\alpha-1}}{\Gamma(\alpha)} \text{ for } \alpha \neq \mu_j, \alpha \neq 1 + \mu_j, j \in \mathbb{Z}, x \in K,$$

where $\mu_j = \frac{2\pi\sqrt{-1}j}{\ln q}$, $j \in \mathbb{Z}$, gives rise to a distribution from $\mathcal{D}'(K)$ called *the one-dimensional Riesz kernel*. This distribution has a meromorphic extension to the whole complex plane α , with poles at the points $\alpha = \mu_j, 1 + \mu_j, j \in \mathbb{Z}$. The distribution $f_0(x)$ is defined by taking

$$f_0(x) = \lim_{\alpha \rightarrow 0} f_\alpha(x) = \delta(x), x \in K \text{ (the Dirac distribution)}$$

where the limit is understood in the weak sense. Notice that $f_0(x) = \lim_{\alpha \rightarrow \mu_j} f_\alpha(x)$ for $j \in \mathbb{Z}$. The definition of the distribution $f_1(x)$ requires to substitute the space of test functions by the Lizorkin space of the first kind, see e.g. [1, Section 9.2]. We do not use this approach in this article.

We recall that if $\alpha \in \mathbb{C}$, with $\text{Re}(\alpha) \neq 1$, then

$$(4.6) \quad \int_K f_\alpha(\xi) \widehat{\phi}(\xi) |d\xi|_K = \int_K |x|_K^{-\alpha} \phi(x) |dx|_K,$$

for $\phi \in \mathcal{D}(K)$ with the convention that $f_0(\xi) = \delta(\xi)$, see e.g. [26, Theorem 4.5].

Remark 3. If $T \in \mathcal{D}'(K^n)$ and $G \in \mathcal{D}'(K^m)$, then its direct product is the distribution defined by the formula

$$(T(x) \times G(y), \varphi) = (T(x), (G(y), \varphi(x, y))) \text{ for } \varphi(x, y) \in \mathcal{D}(K^{n+m}).$$

By using that any test function $\varphi(x, y)$ in $\mathcal{D}(K^{n+m})$ is a linear combination of test functions of the form $\phi_k(x) \psi_k(y)$ with $\phi_k(x) \in \mathcal{D}(K^n)$ and $\psi_k(y) \in \mathcal{D}(K^m)$, one verifies that the Fourier transform of $T \times G$ in $\mathcal{D}'(K^{n+m})$ is given by the formula

$$\widehat{(T \times G, \varphi)} = (\widehat{T} \times \widehat{G}, \varphi) = (T \times G, \widehat{\varphi}) \text{ for } \varphi \in \mathcal{D}(K^{n+m}).$$

For further details, the reader may consult [29, Chap. 1, Sect. VI].

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $x = (x_1, \dots, x_n) \in K^n$. We define $f_\alpha(x) := f_{\alpha_1}(x_1) \times \dots \times f_{\alpha_n}(x_n) \in \mathcal{D}'(K^n)$ as the *multidimensional Riesz kernel*, which is the direct product of the unidimensional Riesz kernels $f_{\alpha_j}(x_j)$, $j = 1, \dots, n$. We will identify the distribution $f_\alpha(x)$ with the function $\prod_{i=1}^n f_{\alpha_i}(x_i)$. Thus, from Remark 3 and (4.6), for $\text{Re}(\alpha_j) \neq 1$ for $j \in \{1, \dots, n\}$,

$$(4.7) \quad \mathcal{F}_{x \rightarrow \xi}(f_\alpha(x)) = \prod_{i=1}^n |\xi_i|_K^{-\alpha_i} \text{ in } \mathcal{D}'(K^n),$$

with the convention $f_0(x) = \delta(x) \in \mathcal{D}'(K^n)$.

4.2.2. *Vladimirov operators.* Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, with $\operatorname{Re}(\alpha_i) \geq 0$ for all i . The multidimensional Vladimirov operator is defined as

$$(4.8) \quad (\mathbf{D}^\alpha \phi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left(\prod_{i=1}^n |\xi_i|_K^{\alpha_i} \mathcal{F}_{x \rightarrow \xi} \phi \right), \quad \alpha \in \mathbb{C}^n, \quad \phi \in \mathcal{D}(K^n).$$

Notice that $\mathbf{D}^\alpha : \mathcal{H}_\infty(K^n) \rightarrow \mathcal{H}_\infty(K^n)$ is continuous operator, cf. Lemma 2. Let $[\cdot, \cdot]$ denote the pairing between \mathcal{H}_∞^* and \mathcal{H}_∞ , see (3.2). Then \mathbf{D}^α has an adjoint operator $\mathbf{D}^{\alpha*} : \mathcal{H}_\infty^*(K^n) \rightarrow \mathcal{H}_\infty^*(K^n)$. Note that

$$(\mathbf{D}^\alpha \phi)(x) = f_{-\alpha}(x) * \phi(x) = (f_{\alpha_1}(y_1) \times \dots \times f_{\alpha_n}(y_n), \phi(x - y)),$$

for $\phi \in \mathcal{D}(K^n)$, in the case $\operatorname{Re}(\alpha_j) \in [0, 1) \cup (1, \infty)$ for $j = 1, \dots, n$.

Proposition 1. (i) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, with $\operatorname{Re}(\alpha_i) > 0$ for all i , $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$, with $\operatorname{Re}(\beta_i) > 0$ for all i . The following formula holds:

$$\left[\mathbf{D}^{\beta*} \mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}, g \right] = \left[\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i} + \overline{\beta_i} - 1} \right\}, g \right]$$

for $g \in \mathcal{H}_\infty(K^n)$ and $\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\} \in \mathcal{H}_\infty^*(K^n)$. In particular, the distribution $\mathcal{F}_{y \rightarrow x} \left[\prod_{i=1}^n |y_i|_K^{-\alpha_i} \right]$ is a \mathcal{H}_∞^* -valued function, which admits an analytic continuation to \mathbb{C}^n .

(ii) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$, with $\operatorname{Re}(\beta_i) > 0$ for all i . The following formula holds:

$$\int_{K^n} f_\alpha(\xi) \widehat{g}(\xi) |d^n \xi|_K = \int_K \prod_{i=1}^n |x_i|_K^{-\alpha_i} g(x) |d^n x|_K,$$

for $g \in \mathcal{H}_\infty(K^n)$. In particular, the distribution $\mathcal{F}_{x \rightarrow \xi} [f_\alpha(x)]$ is a \mathcal{H}_∞^* -valued function, which admits an analytic continuation to \mathbb{C}^n .

Remark 4. The formula given in the second part of Proposition 1 shows that $\mathcal{F}_{x \rightarrow \xi} [f_1(x)]$, with $\mathbf{1} = (1, \dots, 1) \in \mathbb{C}^n$, is a well-defined \mathcal{H}_∞^* -valued function.

Proof. (i) By the existence of $\mathbf{D}^{\beta*}$, we have

$$\left[\mathbf{D}^{\beta*} \mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}, g \right] = \left[\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}, \mathbf{D}^\beta g \right].$$

Now

$$\begin{aligned} \left[\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}, \mathbf{D}^\beta g \right] &= \int_{K^n} \prod_{i=1}^n |x_i|_K^{\alpha_i-1} \widehat{\mathbf{D}^\beta g}(x) |d^n x|_K \\ &= \int_{K^n} \prod_{i=1}^n |x_i|_K^{\alpha_i + \beta_i - 1} \widehat{g}(\xi) |d^n \xi|_K = \left[\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i} + \overline{\beta_i} - 1} \right\}, g \right]. \end{aligned}$$

Now, since $\prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1}$, and then $\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}$, is a \mathcal{H}_∞^* -valued function, which is holomorphic in $\alpha \in \mathbb{C}^n$ in $\operatorname{Re}(\alpha_i) > 0$ for all i , and $\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i} + \overline{\beta_i} - 1} \right\}$ is a \mathcal{H}_∞^* -valued function, which is holomorphic in $\alpha \in \mathbb{C}^n$ in $\operatorname{Re}(\alpha_i) > -\operatorname{Re}(\beta_i)$, $i = 1, \dots, n$, cf. Lemma 5, we conclude that $\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}$ has an analytic extension to the half-plane $\operatorname{Re}(\alpha_i) > -\operatorname{Re}(\beta_i)$, $i = 1, \dots, n$. By using the fact that β is arbitrary, the principle of analytic continuation assures the existence of an analytic continuation of $\mathcal{F} \left\{ \prod_{i=1}^n |x_i|_K^{\overline{\alpha_i}-1} \right\}$ to the whole \mathbb{C}^n .

(ii) By (4.7), we have

$$(4.9) \quad \int_{K^n} f_\alpha(\xi) \widehat{\phi}(\xi) |d^n \xi|_K = \int_{K^n} \prod_{i=1}^n |x_i|_K^{-\alpha_i} \phi(x) |d^n x|_K,$$

for $\phi \in D(K^n)$ and $\operatorname{Re}(\alpha_i) \neq 1$ for $i = 1, \dots, n$. By Lemma 5, the left-hand side of (4.9) defines a \mathcal{H}_∞^* -valued function, which is holomorphic in $\alpha \in \mathbb{C}^n$ in the half-plane

$$\operatorname{Re}(\alpha_i) > 0 \text{ for } i = 1, \dots, n.$$

By switching $\widehat{\phi}$ and ϕ in (4.9) and applying Lemma 5, we obtain that the functional in the right-hand side of (4.9) defines a \mathcal{H}_∞^* -valued function, which is holomorphic in $\alpha \in \mathbb{C}^n$ in the half-plane

$$\operatorname{Re}(\alpha_i) < 1 \text{ for } i = 1, \dots, n.$$

Then both functionals appearing in (4.9) are holomorphic in the open subset

$$(4.10) \quad \operatorname{Re}(\alpha_i) \in (0, 1) \text{ for } i = 1, \dots, n.$$

Now, by the principle of analytic continuation, and the fact that \mathcal{D} is dense in \mathcal{H}_∞ , cf. [32, Lemma 3.4.], formula (4.9) is valid when $f_\alpha(\xi)$ and $\prod_{i=1}^n |x_i|_K^{-\alpha_i}$ are considered \mathcal{H}_∞^* -valued functions of α , with $\operatorname{Re}(\alpha_i) \neq 1$ for all i . Notice that (4.9) can be re-written as

$$[\mathcal{F}_{x \rightarrow \xi} \{f_{\overline{\alpha}}(x)\}, g] = [\mathcal{F}_{y \rightarrow x} \left\{ \prod_{i=1}^n |y_i|_K^{\overline{\alpha}_i} \right\}, \widehat{g}], \quad g \in \mathcal{H}_\infty^*,$$

where the Fourier transforms are understood in \mathcal{D}' . Finally, since $\mathcal{F}_{y \rightarrow x} \left\{ \prod_{i=1}^n |y_i|_K^{\overline{\alpha}_i} \right\}$ admits an analytic continuation to \mathbb{C}^n , we conclude that $\mathcal{F}_{x \rightarrow \xi} \{f_{\overline{\alpha}}(x)\}$ also admits an analytic continuation to \mathbb{C}^n . \square

Remark 5. *It is important to mention that the verification that the functionals appearing in both sides of the formula given in Proposition 1-(ii) have a common domain of regularity is an essential matter. There are several examples of functional equations for local zeta functions where the domain of regularity is ‘the empty set’. For an in-depth discussion of this phenomenon the reader may consult [24, pp. 551-552] and the references therein.*

4.3. Meromorphic continuation of elementary integrals in \mathcal{H}_∞ . We fix $\mathbf{N} = (N_1, \dots, N_n) \in (\mathbb{N} \setminus \{0\})^n$, $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{N} \setminus \{0, 1\})^n$. The elementary integral attached to (\mathbf{N}, \mathbf{v}) and $g \in \mathcal{H}_\infty(K^n)$ is defined as

$$E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v}) = \int_{K^n} \prod_{i=1}^r |\xi_i|_K^{N_i s + v_i - 1} \widehat{g}(\xi) |d^n \xi|_K,$$

where $1 \leq r \leq n$. By Lemma 5, $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ defines \mathcal{H}_∞^* -valued holomorphic function of s in the half-plane $\operatorname{Re}(s) > \max_{1 \leq i \leq n} \frac{-v_i}{N_i}$.

Definition 1. Let $\{\gamma_i\}_{i \in \mathbb{N} \setminus \{0\}}$ be a sequence of positive real numbers such that $\gamma_1 \geq 1$. The generalized arithmetic progression generated by $\{\gamma_i\}_{i \in \mathbb{N}}$ is the sequence $M = \{m_i\}_{i \in \mathbb{N}}$ of real numbers defined as: (1) $m_0 = 0$ and $m_1 = \gamma_1 - 1$; (2) $m_l = \sum_{j=1}^l \gamma_j$ for $l \geq 2$.

Proposition 2. Let $M_j = \{m_i^{(j)}\}_{i \in \mathbb{N}}$, for $j = 1, \dots, n$, be given generalized arithmetic progressions. Then $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ has an analytic continuation the whole complex plane as a $\mathcal{H}_\infty^*(K^n)$ -valued meromorphic function of s , denoted again as $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$, and the real parts of the possible poles of $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ belong to the set $\cup_{i=1}^r \frac{-(v_i + M_i)}{N_i}$. In particular the real parts of the possible poles are negative real numbers.

Proof. First, without loss of generality, we may assume that $r = n$. Indeed, by using the fact that $|\widehat{g}(\xi)| \prod_{i=1}^r |\xi_i|_K^{N_i \operatorname{Re}(s_i) + v_i - 1} \in L^1(K^n, |d^n \xi|_K)$ for $\operatorname{Re}(s) > \max_{1 \leq i \leq r} \frac{-v_i}{N_i}$, and applying Fubini's theorem, we have

$$E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v}) = \int_{K^n} \prod_{i=1}^r |\xi_i|_K^{N_i s_i + v_i - 1} \mathcal{F} \left(\mathbf{P}_{(\xi_{r+1}, \dots, \xi_n), 0} g \right) (\xi_1, \dots, \xi_r) |d^r \xi|_K,$$

with $\mathbf{P}_{(\xi_{r+1}, \dots, \xi_n), 0} g(\xi_1, \dots, \xi_r) \in \mathcal{H}_\infty(K^r)$, cf. Lemma 4. Consequently, we can take $r = n$. By taking $\alpha_i = N_i s + v_i - 1$, $i = 1, \dots, n$, and applying Proposition 1-(i), we have that $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ has an analytic continuation the whole complex plane as a $\mathcal{H}_\infty^*(K^n)$ -valued meromorphic function of s .

We now proceed to describe the possible poles of the analytic continuation of $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$: we start with the integral

$$E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v}) = \left[\mathcal{F} \left[\prod_{i=1}^n |\xi_i|_K^{N_i \overline{s} + v_i - 1} \right], g \right]$$

which holomorphic in $\operatorname{Re}(s) > \max_{1 \leq i \leq n} \frac{-v_i}{N_i}$. Now, we take $\gamma_1 = (\gamma_1^{(1)}, \dots, \gamma_1^{(n)})$ an 'arbitrary vector' in $(\mathbb{R}_+ \setminus \{0\})^n$, and consider the integral

$$\left[\mathcal{D}^{\gamma_1} \mathcal{F} \left[\prod_{i=1}^n |\xi_i|_K^{N_i \overline{s} + v_i - 1} \right], g \right] = \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{N_i s_i + v_i - 1} \widehat{\mathcal{D}^{\gamma_1} g}(\xi) |d^n \xi|_K$$

which holomorphic in $\operatorname{Re}(s) > \max_{1 \leq i \leq n} \frac{-v_i}{N_i}$ since $\mathcal{D}^{\gamma_1} : \mathcal{H}_\infty^* \rightarrow \mathcal{H}_\infty^*$. By using that

$$\begin{aligned} & \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{N_i s_i + v_i - 1} \widehat{\mathcal{D}^{\gamma_1} g}(\xi) |d^n \xi|_K \\ &= \prod_{i=1}^n \Gamma(N_i s + v_i) \int_{K^n} \prod_{i=1}^n f_{N_i s + v_i}(\xi) \widehat{\mathcal{D}^{\gamma_1} g}(\xi) |d^n \xi|_K \\ &= \prod_{i=1}^n \Gamma(N_i s + v_i + \gamma_1^{(i)}) \int_{K^n} \prod_{i=1}^n f_{N_i s + v_i + \gamma_1^{(i)}}(\xi) \widehat{g}(\xi) |d^n \xi|_K, \end{aligned}$$

and using Proposition 1-(ii),

$$\begin{aligned} (4.11) \quad & \prod_{i=1}^n \Gamma(N_i s + v_i) \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{-N_i s - v_i} \mathcal{D}^{\gamma_1} g(\xi) |d^n \xi|_K \\ &= \prod_{i=1}^n \Gamma(N_i s + v_i + \gamma_1^{(i)}) \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{-N_i s - v_i - \gamma_1^{(i)}} g(\xi) |d^n \xi|_K. \end{aligned}$$

We take $\gamma_1^{(i)} = \beta N_i$ for $i = 1, \dots, n$, with $\beta \in \mathbb{N} \setminus \{0\}$ arbitrary, and set $z = -(s + \beta)$, thus $-N_i s - v_i - \gamma_1^{(i)} = N_i z - v_i$ and $-N_i s - v_i = N_i z - v_i + \gamma_1^{(i)}$, therefore formula (4.11) becomes

$$\begin{aligned} (4.12) \quad & \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{N_i z - v_i} g(\xi) |d^n \xi|_K \\ &= \frac{\prod_{i=1}^n \Gamma(-N_i z + v_i - \gamma_1^{(i)})}{\prod_{i=1}^n \Gamma(-N_i z + v_i)} \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{N_i z - v_i + \gamma_1^{(i)}} \mathcal{D}^{\gamma_1} g(\xi) |d^n \xi|_K. \end{aligned}$$

Now the integral in the left-hand side of (4.12) is holomorphic on $\operatorname{Re}(z) > \max_i \frac{-1+v_i}{N_i}$, and the integral in the right-hand side of (4.12) is holomorphic in $\operatorname{Re}(z) > -\beta +$

$\max_i \frac{-1+v_i}{N_i}$, and the factor

$$\frac{\prod_{i=1}^n \Gamma(-N_i z + v_i - \gamma_1^{(i)})}{\prod_{i=1}^n \Gamma(-N_i z + v_i)} = \prod_{i=1}^n \left(\frac{1 - q^{-N_i z + v_i - \gamma_1^{(i)} - 1}}{1 - q^{N_i z - v_i + \gamma_1^{(i)}}} \frac{1 - q^{N_i z - v_i}}{1 - q^{-N_i z + v_i - 1}} \right)$$

gives poles with real parts $\operatorname{Re}(z) = \frac{v_i}{N_i} - \beta$ or $\operatorname{Re}(z) = \frac{v_i}{N_i} - \frac{1}{N_i}$. Therefore, in terms of the variable s , the integral in the right-hand side of (4.11), which is holomorphic in $\operatorname{Re}(s) < -\beta + \max_i \frac{1-v_i}{N_i}$, has a meromorphic continuation in the half-plane $\operatorname{Re}(s) < \max_i \frac{1-v_i}{N_i}$, with possible poles having real parts in the set

$$(4.13) \quad \bigcup_{i=1}^n \frac{-v_i}{N_i} \cup \bigcup_{i=1}^n \frac{-(v_i + \gamma_1^{(i)} - 1)}{N_i}.$$

Therefore, the real parts of the possible poles of

$$\begin{aligned} \left[D^{\gamma_1 * \mathcal{F}} \left[\prod_{i=1}^n |\xi_i|_K^{N_i s + v_i - 1} \right], g \right] &= \int_{K^n} \prod_{i=1}^n |\xi_i|_K^{N_i s + v_i + \gamma_1^{(i)} - 1} \widehat{g}(\xi) |d^n \xi|_K \\ &= E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v} + \gamma_1) \end{aligned}$$

belong to set (4.13). We repeat the calculation starting with $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v} + \gamma_1)$ and $\gamma_2 = (\gamma_2^{(1)}, \dots, \gamma_2^{(n)}) \in (\mathbb{R}_+ \setminus \{0\})^n$, to obtain that the real parts of the possible poles of $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v} + \gamma_1 + \gamma_2)$ belong to the set

$$(4.14) \quad \bigcup_{i=1}^n \frac{-v_i}{N_i} \cup \bigcup_{i=1}^n \frac{-(v_i + \gamma_1^{(i)} - 1)}{N_i} \cup \bigcup_{i=1}^n \frac{-(v_i + \gamma_1^{(i)} + \gamma_2^{(i)} - 1)}{N_i}.$$

By proceeding inductively, we obtain the description of the real parts of the possible poles of $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ announced. Finally, by Lemma 5, $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ is holomorphic in the half-plane $\operatorname{Re}(s) > -1$ and consequently all the real parts of the poles of $E_{\widehat{g}}(s; \mathbf{N}, \mathbf{v})$ are negative real numbers, and thus in (4.14) we have to take $\gamma_1^{(i)} \geq 1$. \square

Remark 6. We notice that there is no canonical way of picking the sequence of vectors $\{\gamma_m\}_{m \in \mathbb{N}}$, this is the reason for Definition 1. On the other hand, integral $\int_K |x|_K^{N s + v - 1} e^{-|x|_K^\alpha} |dx|_K$, with $\alpha > 0$, has a meromorphic continuation to the whole complex plane with possible poles belonging to $-(\frac{v + \alpha N}{N})$.

4.4. Meromorphic Continuation of Local Zeta Functions in \mathcal{H}_∞ . Only in this section, K denotes a non-Archimedean local field of characteristic zero, i.e. K is a finite extension of \mathbb{Q}_p , the field of p -adic numbers. In this section, we show the

existence of a meromorphic continuation for the functional $\widehat{\mathfrak{f}}_K^s$, with $\operatorname{Re}(s) > 0$.

A key ingredient is Hironaka's desingularization theorem (analytic version). For an in-depth discussion of this result as well as, an introduction to the necessary material, the reader may consult [19], [16] and the references therein.

Notation 5. We set $\mathfrak{f}^{-1}(0)_{\text{sing}} := \{\xi \in K^n; \mathfrak{f}(\xi) = \nabla \mathfrak{f}(\xi) = 0\}$. If Π is a K -analytic map, then the pull-back of the differential form $\bigwedge_{1 \leq i \leq n} d\xi_i$ by Π is classically

denoted as $\Pi^* \left(\bigwedge_{1 \leq i \leq n} d\xi_i \right)$, but since we use $*$ in connection with dual space of \mathcal{H}_∞

and the adjoint of an operator, we modify this classical notation as $\Pi^* \left(\bigwedge_{1 \leq i \leq n} d\xi_i \right)$.

Similarly, in the case of function, we use the notation $\phi^* = \phi \circ \Pi$. This notation is used only in this section. We denote by $1_A(x)$ the characteristic function of A .

Theorem 1 (Hironaka [19]). *Let K denote a non-Archimedean local field of characteristic zero and $\mathfrak{f}(\xi)$ a non-constant polynomial in $K[\xi_1, \dots, \xi_n]$. Put $X = K^n$. Then there exist an n -dimensional K -analytic manifold Y , a finite set $\mathcal{E} = \{E\}$ of closed submanifolds of Y of codimension 1 with a pair of positive integers (N_E, v_E) assigned to each E , and a proper K -analytic mapping $\Pi : Y \rightarrow X$ satisfying the following conditions:*

- (i) Π is the composite map of a finite number of monomial transformations each with smooth center;
- (ii) $(\mathfrak{f} \circ \Pi)^{-1}(0) = \cup_{E \in \mathcal{E}} E$ and Π induces a K -bianalytic map

$$(4.15) \quad Y \setminus \Pi^{-1} \left(\mathfrak{f}^{-1}(0)_{\text{sing}} \right) \rightarrow X \setminus \mathfrak{f}^{-1}(0)_{\text{sing}};$$

- (iii) at every point b of Y if E_1, \dots, E_r are all the E in \mathcal{E} containing b with respective local equations y_1, \dots, y_r around b and $(N_{E_i}, v_{E_i}) = (N_i, v_i)$, then there exist local coordinates of Y around b of the form $(y_1, \dots, y_r, y_{r+1}, \dots, y_n)$ such that

$$(4.16) \quad (\mathfrak{f} \circ \Pi)(y) = \varepsilon(y) \prod_{1 \leq i \leq r} y_i^{N_i}, \quad \Pi^* \left(\bigwedge_{1 \leq i \leq n} d\xi_i \right) = \eta(y) \prod_{1 \leq i \leq r} y_i^{v_i-1} \left(\bigwedge_{1 \leq i \leq n} dy_i \right)$$

on some neighborhood of b , in which $\varepsilon(y)$, $\eta(y)$ are units of the local ring \mathcal{O}_b of Y at b .

We call the pair (Y, Π) an embedded resolution of singularities of the map $\mathfrak{f} : K^n \rightarrow K$. The set $\{(N_E, v_E)\}_{E \in \mathcal{E}}$ is called the numerical data of the resolution (Y, Π) .

Remark 7. The following facts will be used later on: (i) Y is a 2-countable and totally disconnected space. This follows from the fact that \mathbb{Q}_p^n is a 2-countable space, and from the fact that Y is obtained by a gluing a finite number of subspaces of \mathbb{Q}_p^n . (ii) There exists a covering of Y of the form $\cup_{m \in \mathbb{N}} U_m$ with each U_m open and closed, $U_i \cap U_j = \emptyset$ if $i \neq j$ and such that in local coordinates each U_m has the form $c_m + (\pi^{e_m} R_K)^n$ with $c_m \in K^n$ and $e_m \in \mathbb{N}$ for each m . (iii) We denote by $\mathcal{D}(Y)$ the space of complex-valued locally constant functions with compact support defined in Y , any such function is a linear combination of characteristic functions of open and compact subsets of Y , see e.g. [16, Chapter 7]. (iv) The differential form $\bigwedge_{1 \leq i \leq n} d\xi_i$ in K^n (considered as K -analytic manifold) induces a measure, denoted as $\left| \bigwedge_{1 \leq i \leq n} d\xi_i \right|_K$, which agrees with normalized Haar measure of K^n , see e.g. [16, Chapter 7].

Theorem 2. Assume that K is a non-Archimedean local field of characteristic zero and let \mathfrak{f} denote an arbitrary element of $R_K[\xi_1, \dots, \xi_n] \setminus R_K$; take $g \in \mathcal{H}_\infty(K^n)$, $s \in \mathbb{C}$, with $\text{Re}(s) > 0$. Then $\left[\widehat{\left| \mathfrak{f} \right|_K^s}, g \right]$ defines a $\mathcal{H}_\infty(K^n)$ -valued holomorphic function

of s , which admits a meromorphic continuation, denoted again as $\left[\widehat{\mathfrak{f}}_K^{\overline{s}}, g\right]$, to the whole complex plane. Furthermore, if $\Pi : Y \rightarrow X$, with $X = K^n$, $\mathcal{E} = \{E\}$ and (N_E, v_E) as in Theorem 1, then the possible real parts of the poles of $\left[\widehat{\mathfrak{f}}_K^{\overline{s}}, g\right]$ are negative real numbers belonging to the set

$$\bigcup_{E \in \mathcal{E}} \frac{-(v_E + M_E)}{N_E},$$

where each M_E is a generalized arithmetic progression.

Proof. The fact that $\left[\widehat{\mathfrak{f}}_K^{\overline{s}}, g\right]$ defines a $\mathcal{H}_\infty(K^n)$ -valued holomorphic function of s in the half-plane $\operatorname{Re}(s) > 0$ follows from Lemma 3. To establish the meromorphic continuation, we pick an embedded resolution of singularities of the map $\mathfrak{f} : K^n \rightarrow K$ as in Theorem 1 and we use all the notation that was introduced there. We take $\phi \in \mathcal{D}(K^n)$ and use (4.15) as an analytic change of variables in $\left[\widehat{\mathfrak{f}}_K^{\overline{s}}, \phi\right]$ to obtain

$$\left[\widehat{\mathfrak{f}}_K^{\overline{s}}, \phi\right] = \int_{X \setminus \mathfrak{f}^{-1}(0)} |\mathfrak{f} \circ \Pi(y)|_K^{\overline{s}} (\phi \circ \Pi)(y) \left| \Pi^\sharp \left(\bigwedge_{1 \leq i \leq n} d\xi_i \right) \right|_K \quad \text{for } \operatorname{Re}(\overline{s}) > 0.$$

At very point $b \in Y$ we can take a chart (V, h_V) with V open and compact and coordinates $(y_1, \dots, y_r, y_{r+1}, \dots, y_n)$ such that formulas (4.16) hold in V . Since $\phi \circ \Pi$, $|\varepsilon(y)|_K$, $|\eta(y)|_K$ are locally constant functions, by subdividing V as a finite union of disjoint open and compact subsets U_m , we have $\phi \circ \Pi|_{U_m} = \phi(\Pi(b))$, $|\varepsilon(y)|_K|_{U_m} = |\varepsilon(b)|_K$, $|\eta(y)|_K|_{U_m} = |\eta(b)|_K$ and further $h_V(U_m) = \tilde{y}_m + \pi^{e_m} R_K^n = B_{-e_m}^n(\tilde{y}_m)$ for some $\tilde{y}_m \in K^n$, $e_m \in \mathbb{N}$. Now, by using the fact that Y is 2-countable we may assume that $\{U_m\}_{m \in \mathbb{N}}$ is a covering of Y consisting of open and compact subsets which are pairwise disjoint. Consequently, we have, first, a map

$$\underline{*} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$$

$$\phi \rightarrow \phi^* := \phi \circ \Pi,$$

in addition, $\phi^*|_{U_m}$ is an element of $\mathcal{D}(B_{-e_m}^n(\tilde{y}_m)) \hookrightarrow \mathcal{D}(K^n)$, where K^n is an affine space with coordinates $(y_1, \dots, y_r, y_{r+1}, \dots, y_n)$; and second,

$$(4.17) \quad \left[\widehat{\mathfrak{f}}(\xi)_K^{\overline{s}}, \phi(\xi)\right] = \sum_{m \in \mathbb{N}} d_m \left[\mathcal{F} \left(\tilde{1}_{B_{-e_m}^n(\tilde{y}_m)}(y) \prod_{i=1}^r |y_i|_K^{N_i \overline{s} + v_i - 1} \right), \phi^*(y) \right]$$

for $\operatorname{Re}(\overline{s}) > 0$, where $\tilde{1}_{B_{-e_m}^n(\tilde{y}_m)}(y)$ is the characteristic function of $\tilde{y}_m + \pi^{e_m} R_K^n$, $1 \leq r = r(n) \leq n$, and $\tilde{1}_{B_{-e_m}^n(\tilde{y}_m)}(y) \phi^*(y)$ is an element of $\mathcal{D}(K^n)$. Since $(\mathcal{D}(K^n), d)$ is dense in (\mathcal{H}_∞, d) and

$$\mathcal{F} \left(\tilde{1}_{B_{-e_m}^n(\tilde{y}_m)}(y) \prod_{i=1}^r |y_i|_K^{N_i \overline{s} + v_i - 1} \right) \in \mathcal{H}_\infty^*(K^n) \quad \text{for } \operatorname{Re}(\overline{s}) > 0, m \in \mathbb{N},$$

then (4.17) extends to an equality between functionals in \mathcal{H}_∞^* in the half-plane $\operatorname{Re}(\overline{s}) > 0$. Then, from Proposition 2 follows that $\left[\widehat{\mathfrak{f}}(\xi)_K^{\overline{s}}\right]$ has a meromorphic

continuation to the whole complex plane as a \mathcal{H}_∞^* -valued function and that the real parts of the possible poles belong to the set

$$\bigcup_{E \in \mathcal{E}} \left\{ \frac{-(v_E + M_E)}{N_E} \right\}.$$

□

Remark 8. In [18, Chapter III, Section 5] Igusa computed an embedded resolution of singularities for a strongly non-degenerate form, with numerical data $\{(1, 1), (d, n)\}$. Then, the Claim in Section 4.1 agrees with Theorem 2.

5. FUNDAMENTAL SOLUTIONS AND LOCAL ZETA FUNCTIONS

Theorem 3. Let \mathfrak{f} be a non-constant polynomial with coefficients in R_K , with K a non-Archimedean local field of arbitrary characteristic. Then, the following assertions are equivalent:

- (i) there exists $E \in \mathcal{H}_\infty^*$ such that $\widehat{E}|_{\mathfrak{f}|_K} = 1$ in L^2 ;
- (ii) set $\mathbf{A}(\partial, \mathfrak{f})g = \mathcal{F}^{-1}(|\mathfrak{f}|_K \mathcal{F}(g))$ for $g \in \text{Dom}(\mathbf{A}(\partial, \mathfrak{f})) := \{g \in L^2; |\mathfrak{f}|_K \widehat{g} \in L^2\}$. There exists $E \in \mathcal{H}_\infty^*$ such that $\mathbf{A}^*(\partial, \mathfrak{f})E = \delta$ in \mathcal{H}_∞^* ;
- (iii) there exists $E \in \mathcal{H}_\infty^*$ such that $E * h \in \mathcal{H}_\infty^*$ for any $h \in \mathcal{H}_\infty$, and $u = E * g$ is a solution of $\mathbf{A}^*(\partial, \mathfrak{f})u = g$ in \mathcal{H}_∞^* , for any $g \in \mathcal{H}_\infty$.

Definition 2. The functional $E \in \mathcal{H}_\infty^*$ is called a fundamental solution for $\mathbf{A}^*(\partial, \mathfrak{f})$.

Proof. (i) \Rightarrow (ii) Since $\mathbf{A}(\partial, \mathfrak{f}) : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ is a continuous operator, cf. Lemma 2, and $E \in \mathcal{H}_\infty^*$,

$$\begin{aligned} [\mathbf{A}^*(\partial, \mathfrak{f})E, g] &= [E, \mathbf{A}(\partial, \mathfrak{f})g] = \int_{K^n} \widehat{\widehat{E}} |\mathfrak{f}|_K \widehat{g} |d^n \xi|_K \\ &= \int_{K^n} \widehat{\widehat{E}} |\mathfrak{f}|_K \widehat{g} |d^n \xi|_K = \int_{K^n} \widehat{g} |d^n \xi|_K = [\delta, g], \end{aligned}$$

for $g \in \mathcal{H}_\infty$.

(ii) \Rightarrow (i) $[\mathbf{A}^*(\partial, \mathfrak{f})E, g] = [\delta, g]$ implies

$$\int_{K^n} \widehat{\widehat{E}} |\mathfrak{f}|_K \widehat{g} |d^n \xi|_K = \int_{K^n} \widehat{g} |d^n \xi|_K,$$

i.e.

$$\int_{K^n} \left\{ \widehat{\widehat{E}} |\mathfrak{f}|_K - 1 \right\} \widehat{g} |d^n \xi|_K = 0$$

for any $g \in \mathcal{H}_\infty$. Since \mathcal{H}_∞ is dense in L^2 , because $\mathcal{D} \hookrightarrow \mathcal{H}_\infty$, the functional $g \rightarrow \int_{K^n} \left\{ \widehat{\widehat{E}} |\mathfrak{f}|_K - 1 \right\} \widehat{g} |d^n \xi|_K$ extends to L^2 as the zero functional, i.e. the function $\widehat{\widehat{E}} |\mathfrak{f}|_K - 1$ is orthogonal to any $\widehat{g} \in L^2$, which implies that $\widehat{\widehat{E}} |\mathfrak{f}|_K = 1$ in L^2 .

(iii) \Rightarrow (i) Take $h \in \mathcal{H}_\infty$, and $u = E * h \in \mathcal{H}_\infty^*$, then

$$[\mathbf{A}^*(\partial, \mathfrak{f})u, h] = [u, \mathbf{A}(\partial, \mathfrak{f})h] = [g, h]$$

i.e.

$$(5.1) \quad \int_{K^n} \widehat{\widehat{E * g}} |\mathfrak{f}|_K \widehat{h} |d^n \xi|_K = \int_{K^n} \widehat{\widehat{E}} \widehat{h} |d^n \xi|_K.$$

By using that $E * h \in \mathcal{D}'(K^n)$, see (3.4), we have $\widehat{\widehat{E * h}} = \widehat{\widehat{E}} \widehat{h}$ in $\mathcal{D}'(K^n)$, and thus (5.1) becomes

$$\int_{K^n} \left\{ \left(\widehat{\widehat{E}} |\mathfrak{f}|_K - 1 \right) \widehat{g} \right\} \widehat{h} |d^n \xi|_K = 0,$$

which implies that $(\widehat{E}|f|_K - 1)\widehat{g} = 0$ in L^2 for any $g \in \mathcal{H}_\infty$, and hence $\widehat{E}|f|_K = 1$ in L^2 .

(ii) \Rightarrow (iii) First $E * g$ exists in $\mathcal{D}'(K^n)$ if and only if $\widehat{E}\widehat{g} \in \mathcal{D}'(K^n)$, see e.g. [29, p.115]. We check this last condition: taking $\theta \in \mathcal{D}(K^n)$, we have for some non-negative integer l that

$$\begin{aligned} \left| \int_{K^n} \widehat{E}\widehat{g}\theta |d^n\xi|_K \right| &= \left| \int_{K^n} \widehat{E}\mathcal{F}(g * \mathcal{F}^{-1}(\theta)) |d^n\xi|_K \right| \leq \|E\|_{-l} \|g * \mathcal{F}^{-1}\theta\|_l \\ &\leq \|\theta\|_{L^\infty} \|E\|_{-l} \|g\|_l. \end{aligned}$$

This shows that $E * g \in \mathcal{D}'(K^n)$ and that $\widehat{E * g} = \widehat{E}\widehat{g}$ in $\mathcal{D}'(K^n)$. On the other hand, $E * g \in \mathcal{H}_\infty^*(K^n)$ for any $g \in \mathcal{H}_\infty(K^n)$, because

$$\|E * g\|_{-l}^2 = \int_{K^n} [\xi]_K^{-l} \left| \widehat{E} \right|^2 |\widehat{g}|^2 |d^n\xi|_K \leq \|\widehat{g}\|_{L^\infty}^2 \|E\|_{-l}^2$$

for any positive integer l , since $\widehat{g} \in C_0(K^n)$, cf. Lemma 1-(v). Now by using that (i) \Leftrightarrow (ii), we have

$$\begin{aligned} [\mathbf{A}^*(\partial, f)u, h] &= [E * g, \mathbf{A}(\partial, f)h] = \int_{K^n} \widehat{E}\widehat{g}|f|_K \widehat{h} |d^n\xi|_K \\ &= \int_{K^n} \overline{\widehat{E}|f|_K} \widehat{g}\widehat{h} |d^n\xi|_K = \int_{K^n} \widehat{g}\widehat{h} |d^n\xi|_K = [g, h]. \end{aligned}$$

□

Theorem 4. *Let f be a non-constant polynomial with coefficients in R_K , with K a non-Archimedean local field of arbitrary characteristic. Assume that $\left[\widehat{f|_K}^s, g \right]$ has a meromorphic continuation to the whole complex plane as a $\mathcal{H}_\infty^*(K^n)$ -valued function of s , with poles having negative real parts. Then there exists a fundamental solution for operator $\mathbf{A}^*(\partial, f)$.*

Proof. The proof is based in the Gel'fand-Shilov method of analytic continuation, see [16, p. 65-67]. By the hypothesis that $\left[\widehat{f|_K}^s, g \right]$ has an analytic continuation to the whole complex plane, there exists a Laurent expansion around $s = -1$ of the form

$$\left[\widehat{f|_K}^s, g \right] = \int_{K^n \setminus f^{-1}(0)} |f|_K^s \widehat{g} |d^n\xi|_K = \sum_{k \in \mathbb{Z}} [T_k, g] (s+1)^k$$

where $T_k \in \mathcal{H}_\infty^*$ for $k \in \mathbb{Z}$. This fact is established by using the ideas presented in [16, p. 65-67]. Now,

$$\begin{aligned} \left[\widehat{f|_K}^s, \mathbf{A}(\partial, f)g \right] &= \int_{K^n \setminus f^{-1}(0)} |f|_K^{s+1} \widehat{g} |d^n\xi|_K = \sum_{k \in \mathbb{Z}} [T_k, \mathbf{A}(\partial, f)g] (s+1)^k \\ &= [T_0, \mathbf{A}(\partial, f)g] + \sum_{k=1}^{\infty} [T_k, \mathbf{A}(\partial, f)g] (s+1)^k, \end{aligned}$$

since $\int_{K^n \setminus \mathfrak{f}^{-1}(0)} |\mathfrak{f}|_K^{s+1} \widehat{g} |d^n \xi|_K$ does not have poles with real part -1 . Therefore

$$\begin{aligned} \lim_{s \rightarrow -1} \left[\widehat{|\mathfrak{f}|_K^s}, \mathbf{A}(\partial, \mathfrak{f})g \right] &= \int_{K^n \setminus \mathfrak{f}^{-1}(0)} |\mathfrak{f}|_K^{s+1} \widehat{g} |d^n \xi|_K = [T_0, \mathbf{A}(\partial, \mathfrak{f})g] \\ &= \int_{K^n \setminus \mathfrak{f}^{-1}(0)} \widehat{g} |d^n \xi|_K, \end{aligned}$$

since $\widehat{g} \in L^1$, i.e. $[T_0, \mathbf{A}(\partial, \mathfrak{f})g] = [\delta, g]$, for any $g \in \mathcal{H}_\infty$, which implies that $\mathbf{A}^*(\partial, \mathfrak{f})T_0 = \delta$ with $T_0 \in \mathcal{H}_\infty^*$. \square

REFERENCES

- [1] Albeverio S., Khrennikov A. Yu., Shelkovich V. M., Theory of p -adic distributions: linear and nonlinear models. Cambridge University Press, 2010.
- [2] Arnold V. I., Gussein-Zade S. M. and Varchenko A. N., Singularités des applications différentiables, Vol II. Éditions Mir, Moscou, 1986.
- [3] Atiyah M. F., Resolution of singularities and division of distributions, Comm. Pure Appl. Math. 23 (1970), 145–150.
- [4] Belkale P. and Brosnan P., Periods and Igusa local zeta functions, Int. Math. Res. Not. 49 (2003), 2655–2670.
- [5] Berthelot Pierre, Introduction à la théorie arithmétique des \mathcal{D} -modules. Cohomologies p -adiques et applications arithmétiques, II, Astérisque No. 279 (2002), 1–80.
- [6] Bernstein I. N., Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients, Functional Analysis and its Applications 5, No.2, 1-16 (1972).
- [7] Bollini C.G., Giambiagi J. J., González Domínguez A., Analytic regularization and the divergencies of quantum field theories, Il Nuovo Cimento XXXI, no. 3 (1964) 550-561.
- [8] Bocardo-Gaspar Miriam, García-Compeán H., Zúñiga-Galindo W. A., Regularization of p -adic String Amplitudes, and Multivariate Local Zeta Functions. arXiv:1611.03807.
- [9] Cassaigne Julien, Maillot Vincent, Hauteur des hypersurfaces et fonctions zêta d'Igusa, J. Number Theory 83 (2000), no. 2, 226–255.
- [10] Denef J., Report on Igusa's Local Zeta Function, Séminaire Bourbaki 43 (1990-1991), exp. 741; Astérisque 201-202-203 (1991), 359–386.
- [11] Denef J. and Loeser F., Motivic Igusa zeta functions, J. Alg. Geom. 7 (1998), 505–537.
- [12] Gel'fand I. M., Shilov G. E., Generalized Functions. Vol. 1. Properties and operations. AMS Chelsea publishing, 2010.
- [13] Gel'fand I. M., Shilov G. E., Generalized functions. Vol. 2. Spaces of fundamental and generalized functions. AMS Chelsea publishing, 2010.
- [14] Gel'fand I. M., Vilenkin N. Ya, Generalized functions. Vol. 4. Applications of harmonic analysis. AMS Chelsea publishing, 2010.
- [15] Hida Takeyuki, Kuo Hui-Hsiung, Potthoff Jürgen, Streit, Ludwig, White noise. An infinite-dimensional calculus. Kluwer Academic Publishers Group, 1993.
- [16] Igusa J.-I., An introduction to the theory of local zeta functions. AMS/IP Studies in Advanced Mathematics, 2000.
- [17] Igusa J.-I., A stationary phase formula for p -adic integrals and its applications, Algebraic geometry and its applications, Springer-Verlag (1994), pp. 175–194.
- [18] Igusa J.-I., Forms of higher degree. The Narosa Publishing House, 1978.
- [19] Hironaka H., Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. Math. 79 (1964), 109–326.
- [20] Kochubei Anatoly N., Pseudo-differential equations and stochastics over non-Archimedean fields. Marcel Dekker, 2001.
- [21] Loeser F., Fonctions zêta locales d'Igusa à plusieurs variables, intégration dans les fibres, et discriminants, Ann. Sc. Ec. Norm. Sup. 22 (1989), no. 3, 435–471.
- [22] Mustață Mircea, Bernstein-Sato polynomials in positive characteristic, J. Algebra 321 (2009), no. 1, 128–151.

- [23] Obata Nobuaki, White noise calculus and Fock space. Lecture Notes in Mathematics, 1577. Springer-Verlag, 1994.
- [24] Rubin Boris, Riesz potentials and integral geometry in the space of rectangular matrices, Adv. Math. 205 (2006), no. 2, 549–598.
- [25] Speer Eugene R., Generalized Feynman amplitudes, Annals of Mathematics Studies, No. 62. Princeton University Press, 1969.
- [26] Taibleson M. H., Fourier analysis on local fields. Princeton University Press, 1975.
- [27] Veys W. and Zúñiga-Galindo W.A., Zeta functions for analytic mappings, log-principalization of ideals and Newton polyhedra, Trans. Amer. Math. Soc. 360 (2008), 2205–2227.
- [28] Veys W. and Zúñiga-Galindo W.A., Zeta functions and oscillatory integrals for meromorphic functions, Adv. Math. <http://dx.doi.org/10.1016/j.aim.2017.02.022>.
- [29] Vladimirov V. S., Volovich I. V., Zelenov E. I., P-adic analysis and mathematical physics. World Scientific, 1994.
- [30] Weil André, Basic number theory. Springer-Verlag, 1967.
- [31] Yosida Kōsaku, Functional analysis. Springer-Verlag, 1965.
- [32] Zúñiga-Galindo W. A., Non-Archimedean white noise, pseudodifferential stochastic equations, and massive Euclidean fields, J. Fourier Anal. Appl. 23, no 2 (2017) 288–323.
- [33] Zúñiga-Galindo W. A., Pseudodifferential equations over non-Archimedean spaces. Lectures Notes in Mathematics 2174, Springer, 2016.
- [34] Zuniga-Galindo W. A., Local zeta functions and Newton polyhedra, Nagoya Math. J. 172 (2003), 31–58.
- [35] Zuniga-Galindo W. A., Fundamental solutions of pseudo-differential operators over p -adic fields, Rend. Sem. Mat. Univ. Padova 109 (2003), 241–245.
- [36] Zúñiga-Galindo W. A., Igusa’s local zeta functions of semiquasihomogeneous polynomials, Trans. Amer. Math. Soc. 353 (2001), no. 8, 3193–3207.

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL, DEPARTAMENTO DE MATEMÁTICAS, UNIDAD QUERÉTARO, LIBRAMIENTO NORPONIENTE #2000, FRACC. REAL DE JURIQUILLA. SANTIAGO DE QUERÉTARO, QRO. 76230, MÉXICO.

E-mail address: `wazuniga@math.cinvestav.edu.mx`